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Uniform Convergence to the Spectral Radius and Some Related Properties in Banach Algebras

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Introduction

Gelfand's assertion of the equality of the spectral radius of an element a of a Banach algebra with the limit of the sequence $\|a^n\|^{1/n}$ is ample motivation for the study of Banach algebras. In this work, whose scope is the general structure theory of Banach algebras, we investigate conditions of uniformity in this convergence.

In the first chapter we make our fundamental definitions, establish some notation and describe the background to our discussions. The reader should note that definitions are not exclusively made in the first chapter, but as needed. We trust that the index will provided adequate reference.

We follow the definitions with a discussion of stability and then a detailed treatment of the properties of spectral uniformity and topologically bounded index. We conclude by investigating properties which are related, injectivity in particular.

As with most Banach algebra theory we use results from several areas of mathematics; spectral uniformity requires a quantity of calculus and classical analysis, topologically bounded index some post-war ring theory. Where possible we refer to well-known textbooks for proofs of results that we use, but inevitably some are only to be found in more inaccessible sources.

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Contents

1. Definitions

In this chapter we fix some notation and provide motivation for our definitions. We begin with the Banach-algebraic context.

By an *algebra* we shall mean an associative linear algebra, always with the complex field $\mathbb{C}$ as scalars. An algebra A is a *normed algebra* if it possesses a norm $\|\cdot\|$ which is an *algebra norm* i.e.

$$\|ab\| \leq \|a\| \, \|b\| \quad (a, \, b \in A)$$

and when such a norm induces a complete metric, A is a *Banach algebra*. Our normed algebras need not be commutative or possess multiplicative units unless explicitly stated. However, if a normed algebra does have a unit we will assume that it is of unit norm.

When an algebra A has a unit we will write $\mathrm{inv}(A)$ for the group of invertible elements in A and $\mathrm{sing}(A)$ for the singular elements of A. We recall that a *left (right) quasi-inverse* of an element a in an algebra A is $b \in A$ with

$$b \circ a := a + b - ba = 0 \quad (a \circ b = 0).$$

An element $b \in A$ which is both a left and a right quasi-inverse for a is a *quasi-inverse* and a is then *quasi-invertible*. We then use the obvious notation of $\text{q-inv}(A)$ and $\text{q-sing}(A)$.

For an algebra A and $a \in A$ the *spectrum* of a, denoted $\sigma_A(a)$, is given by

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \, \lambda 1 - a \in \mathrm{sing}(A)\}$$

if A has a unit and

$$\sigma_A(a) = \left\{\lambda \in \mathbb{C} : \, \lambda^{-1} a \in \text{q-sing}(A)\right\}$$

otherwise. Where there is no ambiguity as to the algebra in question we will write $\sigma(a)$ for $\sigma_A(a)$. When the spectrum is non-empty the *spectral radius* is $\sup\{|\lambda| : \, \lambda \in \sigma_A(a)\}$. That the spectrum of an element of a normed algebra *is* non-empty was shown by Gelfand in his foundational paper [21] along with the following.

Theorem 1.1. *If a is an element of a normed algebra A then*

$$\inf_{n \in \mathbb{N}} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n},$$

in particular the limit exists. The limit is no greater than the spectral radius of a and there is equality if A is a Banach algebra.

The equality when A is a Banach algebra is the celebrated *spectral radius formula* described by Rudin in [45] as having

> the remarkable feature ... [that it] ... asserts the equality of certain quantities which arise in entirely different ways.

We can, then, view the spectral radius formula as a link between the algebraic and topological in Banach algebras and we are motivated to investigate this link — by seeking to determine constraints on the structure of an algebra imposed by the existence of an algebra norm or complete algebra norm satisfying additional conditions of uniformity.

For a normed algebra A we will use the notation

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \quad (a \in A)$$

and ask when the convergence of $\|a^n\|^{1/n}$ to $r(a)$ is uniform. This question suggests two interpretations: firstly we could consider the uniformity of the convergence of

$$\frac{\|a^n\|^{1/n} - r(a)}{\|a\|}$$

to zero over non-zero $a \in A$. Secondly there is the question of the convergence of $\|a^n\|^{1/n}/r(a)$ to one over the $a \in A$ that have $r(a) > 0$. We will treat both of these properties but concentrate on the former as it seems an intuitively simpler concept by dint of its globality. To this end we introduce the following fundamental quantity.

Definition 1.2. For a subset B of a normed algebra A let

$$V_B(n) = \sup\left\{\|b^n\|^{1/n} - r(b) : \, b \in B, \, \|b\| \le 1\right\} \quad (n \in \mathbb{N})$$

and write V_B for the infimum over $n \in \mathbb{N}$ of $V_B(n)$.

Note that

$$V_B(n) = \sup\left\{\frac{\|b^n\|^{1/n} - r(b)}{\|b\|} : \, b \in B\backslash\{0\}\right\}$$

whenever B is a *cone:* a subset closed under multiplication by non-negative real scalars.

Definition 1.3. We will say that a normed algebra A is *spectrally uniform* if $V_A(n) \to 0$ as $n \to \infty$.

To describe our next definition we will need to review some basic concepts in algebra. An ideal in an algebra A is *left quasi-invertible* if all its elements are left quasi-invertible and the *Jacobson radical* of A, denoted J-rad(A), is the ideal which is the union (in the algebra sense) of all such ideals in A. An algebra A is *semisimple* if J-rad$(A) = \{0\}$ and *radical* if J-rad$(A) = A$.

The *nilpotent* elements of an algebra A are those for which there is some $n \in \mathbb{N}$ with $a^n = 0$. We write $N(A)$ for the set of nilpotent elements of A and say that an ideal I of A is *nil* if it consists only of nilpotent elements (so that A is nil if $A = N(A)$). When A is a normed algebra we say that $a \in A$ is *topologically nilpotent* if $\|a^n\|^{1/n} \to 0$ as $n \to \infty$, i.e. if $r(a) = 0$, and denote the set of such elements $T(A)$. This set is intricately associated with the Jacobson radical when A is a Banach algebra, as may be seen from the raft of characterizations of J-rad(A) conveniently summarised in [4, Theorem 5.3.1]. The following facts are less deep, but since they must be borne in mind throughout this work we present them as a theorem. The proofs may be found in [8, §25 Prop. 1, §17 Th. 7].

Theorem 1.4. *If A is a normed algebra then* J-rad$(A) \subseteq T(A)$*. If A is a Banach algebra then each ideal I of A with $I \subseteq T(A)$ is contained in* J-rad(A) *and if A is also commutative we have* J-rad$(A) = T(A)$*.*

We hope that this provides some justification for considering the uniformity of the convergence of $\|a^n\|^{1/n}$ over those $a \in T(A)$ with unit norm. Such uniformity can also be seen as a topological analogue of a property of interest in the theory of rings. A ring A is of *bounded index* if there is $N \in \mathbb{N}$ such that $a^N = 0$ for all $a \in N(A)$. Such rings have been studied by, amongst others, Jacobson [32] and Klein [35].

Definition 1.5. A normed algebra is of *topologically bounded index* if $V_{T(A)}(n) \to 0$ as $n \to \infty$.

In the case that A is a normed algebra with $T(A) = A$ our definitions of topologically bounded index and spectral uniformity coincide with that for *uniform topological nillity* considered in [18] and [19]. So when looking at normed algebras of topologically bounded index we will, in the main, restrict our attentions those which are not radical.

For a commutative Banach algebra A we have $T(A) =$ J-rad(A) and so in this case A is of topologically bounded index if, and only if, its radical is uniformly topologically nil. Such algebras have been investigated by Dixon & Willis in [20] and in a little-known paper of Gorin & Lin [23] on the Wedderburn decomposition of certain commutative Banach algebras. We will use the main result of the latter in a discussion of the stability of spectral uniformity.

With the bulk of our notation in place we conclude this chapter with a lemma which will simplify several subsequent arguments. It also illustrates a recurrent theme in the topic: that *most* of what can be shown for topologically bounded index can also be shown for spectral uniformity, provided we add some real analysis.

Lemma 1.6. *For a normed algebra A the limits as $n \to \infty$ of $V_A(n)$ and $V_{T(A)}(n)$ exist, and the equalities*

$$\lim_{n\to\infty} V_A(n) = V_A \qquad \lim_{n\to\infty} V_{T(A)}(n) = V_{T(A)}$$

hold.

Proof. To obtain the first equality we modify the argument in [8, §2 Prop. 8]. Let $\epsilon > 0$ be given and choose $k \in \mathbb{N}$ such that

$$\left\|a^k\right\|^{1/k} - r(a) \leq V_A + \epsilon/2 \quad (a \in A,\ \|a\| \leq 1). \tag{1.1}$$

For any $n \in \mathbb{N}$ we may write $n = p(n)k + q(n)$ where $p(n)$, $q(n) \in \mathbb{N}$ and $q(n) \leq k$, and then

$$\|a^n\|^{1/n} \leq \left(\left\|a^k\right\|^{1/k}\right)^{p(n)k/n} \qquad (a \in A,\ \|a\| \leq 1,\ n \in \mathbb{N}).$$

Since $p(n)k/n$ is a sequence of rationals which converge from below to one, we can find (using a straightforward argument in calculus) some n_0 with

$$t^{p(n)k/n} \leq t + \epsilon/2 \quad (t \in [0,1],\ n \geq n_0).$$

We then have that

$$\|a^n\|^{1/n} \leq \left\|a^k\right\|^{1/k} + \epsilon/2 \quad (\|a\| \leq 1,\ n \geq n_0)$$

which, combined with (1.1), and taking the supremum over all $a \in A$ with $\|a\| \leq 1$ shows that $V_A(n) \leq V_A + \epsilon$ for all $n \geq n_0$.

The second equality can be treated similarly, but in this case the fact that

$$V_{T(A)}(n+m)^{n+m} \leq V_{T(A)}(n)^n V_{T(A)}(m)^m \quad (n,\ m \in \mathbb{N})$$

means that we can apply the argument in [8, §2 Prop. 8] directly. ∎

2. Stability

2.1. General Remarks

In this chapter we consider whether topologically bounded index and spectral uniformity are preserved during common constructions used in Banach algebra theory.

There are a number of difficulties involved in showing such stability. In passing to the construction we must be able to control several quantities simultaneously — if a is an element of the construction then we need to find a lower bound for $\|a^n\|^{1/n}$ and upper bounds for $\|a\|$ and $r(a)$ in terms of these quantities for elements in the original algebra. Control of these quantities may not be possible, as is illustrated by Example 2.4.4 which is a normed algebra A with $\overline{T(A)}$ properly contained in $T\left(\overline{A}\right)$. However we can find some results on stability, in some cases when extra conditions are satisfied. We begin by showing that the uniform convergence that we study depends on the topology of the algebra rather than its metric.

We use the convention that an *isomorphism* of normed algebras is an isomorphism of the algebraic structure (an *algebra isomorphism*) and a homeomorphism. When A and B are isomorphic normed algebras we shall write $A \cong B$ and $A \overset{1}{\cong} B$ if the isomorphism is an isometry. We also use the notation $A \hookrightarrow B$ to indicate that A is isomorphic, as a Banach space, to a subspace of B.

Proposition 2.1.1. *Both topologically bounded index and spectral uniformity are preserved under isomorphisms of normed algebras.*

Proof. If A and B are normed algebras and $\psi : A \to B$ is an isomorphism then there is some $C > 0$ with

$$C^{-1}\|a\| \le \|\psi(a)\| \le C\|a\| \quad (a \in A).$$

Then for $a \in A$ we have $r(a) = r(\psi(a))$ and so

$$\begin{aligned} \frac{\|\psi(a)^n\|^{1/n} - r(\psi(a))}{\|\psi(a)\|} &\le C\left(\frac{C^{1/n}\|a^n\|^{1/n} - r(a)}{\|a\|}\right) \\ &\le C\left(\left(C^{1/n} - 1\right) + \frac{\|a^n\|^{1/n} - r(a)}{\|a\|}\right). \end{aligned} \tag{2.1}$$

Taking suprema over all $a \in A$ in (2.1) and letting $n \to \infty$ shows that $V_B \leq C\, V_A$ and so, by symmetry, our claim holds for spectral uniformity. The case of topologically bounded index also follows from (2.1) once we note that $T(B) = \psi((T(A))$. $\square$

2.2. Quotients

For a Banach algebra A and a closed ideal $I \lhd A$, the Banach algebra A/I is the algebra of equivalence classes $[a]_I$ $(a \in A)$ with norm

$$\|[a]_I\| = \inf_{b \in I} \|a + b\| .$$

The following example is of a Banach algebra A of topologically bounded index with an ideal I such that A/I is not of topologically bounded index.

Example 2.2.1. Let A_0 denote the algebra over $\mathbb{C}$ with generators a_n $(n \in \mathbb{N})$ and relations $a_n a_m = 0\,(n \neq m)$. Thus a typical element $x \in A_0$ is of the form

$$x = \sum_{n,m \in \mathbb{N}} \lambda_{n,m} a_n^m \tag{2.2}$$

where only finitely many of the $\lambda_{n,m} \in \mathbb{C}$ are non-zero. With the ℓ^1 norm,

$$\|x\| = \sum_{n,m \in \mathbb{N}} |\lambda_{n,m}|, \tag{2.3}$$

A_0 is a normed algebra the completion, denoted A, can be identified with the algebra of possibly infinite sums (2.2) with the sum (2.3) finite.

If $x \in A$ is non-zero then we can find n, m such that

$$\lambda_{n,1} = \lambda_{n,2} = \cdots = \lambda_{n,m-1} = 0 \neq \lambda_{n,m}$$

so that for each k the coefficient of a_n^{km} in x^k is exactly $\lambda_{n,m}^k$. Then $\|x^k\| \geq |\lambda_{n,m}^k|$ so $r(x) \geq |\lambda_{n,m}| > 0$ and consequently A is vacuously of topologically bounded index.

Now write $I = \overline{\operatorname{span}}\,\{a_n^m : 1 \leq n < m\}$, which is a closed ideal. Then $[a_n]_I^{n+1} = 0$ for $n \in \mathbb{N}$ while

$$\|[a_n]_I^n\| = \|[a_n^n]_I\| = \|a_n^n\| = 1 = \|[a_n]_I\| .$$

It follows that $V_{A/I}(n) = 1$ $(n \in \mathbb{N})$ and so A/I is not of topologically bounded index.

There is one case in which, for particular algebras, topologically bounded index is preserved when we take a quotient — when the ideal in question is the radical, and this is discussed in Section 4.2.

We now turn to the spectral uniformity of quotients. If X is a compact Hausdorff space we shall write $C(X)$ for the Banach algebra of continuous functions on X. The algebraic operations are pointwise and the norm is the supremum norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \quad (f \in C(X)).$$

A *uniform algebra* is a closed subalgebra of some $C(X)$ and a Q-*algebra* is a (commutative) Banach algebra isomorphic to the quotient of a uniform algebra by a closed ideal.

For each element f of $C(X)$ we have $r(f) = \|f\|_\infty$ so that any uniform algebra A is spectrally uniform — one might say that it is as spectrally uniform as it is possible to be, since $V_A(n) = 0$ $(n \in \mathbb{N})$. However it is not difficult to show that the following radical Q-algebra, constructed by Dixon in [14], is not spectrally uniform.

Example 2.2.2. Let $\mathscr{A}(\triangle)$ denote the disc algebra: the algebra of complex continuous functions on the closure of the unit disc $\triangle$ of $\mathbb{C}$ which are analytic on $\triangle$. With pointwise algebraic operations and the supremum norm $\mathscr{A}(\triangle)$ is a uniform algebra.

Let
$$M = \{f \in \mathscr{A}(\triangle) : f(0) = 0\}$$
so that for each $n = 1, 2, \ldots$ the set

$$M^n = \operatorname{span}\{f_1 \cdots f_n : f_1, \ldots, f_n \in M\}$$

is a closed ideal of M. Note that M/M^n is a Q-algebra which is nil of index $n-1$. Let
$$A = c_0\text{-}\bigoplus_{n=2}^{\infty} M/M^n$$
be the algebra of sequences (b_n) with $b_n \in M/M^n$ $(n = 2, 3, \ldots)$ and $\|b_n\| \to 0$ as $n \to \infty$. The norm on A is the supremum norm

$$\|(b_n)\|_\infty = \sup_{n\geq 2} \|b_n\| \quad \big((b_n) \in A\big)$$

and the algebraic operations are pointwise. It is shown in [14] that A is a radical Q-algebra; we now show that it is not spectrally uniform.

We shall let $[f]$ denote the equivalence class in M/M^{n+1} of the function given by $f(z) = z$ $(z \in \overline{\triangle})$ and suppose that g is a function in M^{n+1} so that

$$g(z) = \sum_{k=1}^{\infty} \alpha_k z^{n+k} \quad (z \in \overline{\triangle})$$

for some $\alpha_k \in \mathbb{C}$. Then with $\partial\triangle$ denoting the boundary of $\triangle$

$$\begin{aligned}
\|f^n + g\|_\infty &= \sup_{z\in\triangle}\left|z^n + \sum_{k=1}^{\infty}\alpha_k z^{n+k}\right| \\
&= \sup_{z\in\partial\triangle}\left|1 + \sum_{k=1}^{\infty}\alpha_k z^{k}\right| \\
&= \sup_{z\in\triangle}\left|1 + \sum_{k=1}^{\infty}\alpha_k z^{k}\right| \\
&\geq 1
\end{aligned}$$

by two applications of the maximum modulus principle. This shows that we have $\|[f]^n\|^{1/n} \geq 1$ which implies equality since $\|[f]\| = 1$. Finally, let $a_n \in A$ be the sequence with $[f]$ in the $n+1$-th co-ordinate and zero in the others. We have

$$\|a_n^n\|_\infty^{1/n} = \|[f]^n\|^{1/n} = 1$$

while $\|a_n\|_\infty = 1$ and since $a_n^{n+1} = 0$ we have $V_A(n) = 1$ ($n \in \mathbb{N}$). Thus A is not spectrally uniform.

2.3. Sums & Products

We obtain some information here on various sums and products of Banach algebras. The following propositions are straightforward and also hold when 'topologically bounded index' replaces 'spectrally uniform' in their statements. Since the proofs, in this case, are much simpler we omit the details.

If A is a Banach algebra then subalgebras $A_1, \ldots, A_k$ are *orthogonal* if

$$a_i a_j = 0 \quad (a_i \in A_i,\, a_j \in A_j,\, i \neq j).$$

Proposition 2.3.1. *If a Banach algebra A is the direct sum of orthogonal spectrally uniform closed subalgebras $A_1, \ldots, A_k$, then it is spectrally uniform.*

Proof. Application of the open mapping theorem to

$$\begin{aligned}
A_1 \times \cdots \times A_k &\longrightarrow A \\
(a_1, \ldots, a_k) &\longmapsto a_1 + \cdots + a_k
\end{aligned}$$

(where the Cartesian product has the max-norm) shows that there is a constant $C > 1$ such that

$$\max_{i=1,\ldots,k} \|a_i\| \leq C\,\|a_1 + \cdots + a_k\| \quad (a_j \in A_j,\, j = 1, \ldots, k).$$

So for $a \in A$ with $a = a_1 + \cdots + a_k$ we have

$$C^{-1/n} \max_{i=1,\ldots,k} \|a_i^n\|^{1/n} \leq \|a_1^n + \cdots + a_k^n\|^{1/n} \leq k^{1/n} \max_{i=1,\ldots,k} \|a_i^n\|^{1/n}$$

and so $r(a) = \max_{i=1,\ldots,k} r(a_i)$. Then

$$\begin{aligned}
\|a^n\|^{1/n} - r(a) &\leq \left(k \max_{i=1,\ldots,k} \|a_i^n\|\right)^{1/n} - \max_{i=1,\ldots,k} r(a_i) \\
&\leq \max_{i=1,\ldots,k} \left(k^{1/n} \|a_i^n\|^{1/n} - r(a_i)\right) \\
&\leq \max_{i=1,\ldots,k} \left(V_{A_i}(n) \|a_i\| + (k^{1/n} - 1) \|a_i^n\|^{1/n}\right) \\
&\leq \left(\max_{i=1,\ldots,k} V_{A_i}(n) + (k^{1/n} - 1)\right) \max_{i=1,\ldots,k} \|a_i\|
\end{aligned}$$

and so

$$V_A(n) \leq C \left(\max_{i=1,\ldots,k} V_{A_i}(n) + (k^{1/n} - 1)\right) \tag{2.4}$$

provides an appropriate bound. □

Cartesian products of spectrally uniform Banach algebras satisfy the conditions of Proposition 2.3.1, but in this case we can say a little more.

Proposition 2.3.2. *Suppose that the Banach algebra A is the Cartesian product of Banach algebras A_λ $(\lambda \in \Lambda)$. Then*

$$V_A(n) = \sup_{\lambda \in \Lambda} V_{A_\lambda}(n).$$

Proof. Using an obvious notation we take $a = (a_\lambda)_{\lambda \in \Lambda}$ in A so that

$$\begin{aligned}
\|a^n\|^{1/n} - r(a) &= \sup_{\lambda \in \Lambda} \|a_\lambda^n\|^{1/n} - \sup_{\lambda \in \Lambda} r(a_\lambda^n) \\
&\leq \sup_{\lambda \in \Lambda} \left(\|a_\lambda^n\|^{1/n} - r(a_\lambda^n)\right) \\
&\leq \sup_{\lambda \in \Lambda} V_{A_\lambda}(n) \|a_\lambda\| \\
&\leq \left(\sup_{\lambda \in \Lambda} V_{A_\lambda}(n)\right) \|a\| .
\end{aligned}$$

Hence $V_A(n) \leq \sup_{\lambda \in \Lambda} V_{A_\lambda}(n)$, and the reverse inequality is obvious. □

For tensor products (described in detail in Section 5.1) there seems to be little that we can say. There are commutative semisimple Banach algebras A, B such that the projective tensor product $A \widehat{\otimes} B$ is not semisimple. (This has been shown by Milne using a construction of Enflo; see [8, §43] for a discussion of this fact and of tensor products in general). In particular this example has $T(A\widehat{\otimes}B)$ non-trivial — so in passing to a tensor product we may 'lose control of the spectral radius'.

However we can say something about algebras, the tensor product of which is a spectrally uniform Banach algebra. Recall that a cross-norm on the tensor product $A \otimes B$ is one satisfying

$$\|a \otimes b\| = \|a\| \, \|b\| \quad (a \in A, \, b \in B).$$

Proposition 2.3.3. *Let A and B be Banach algebras and suppose that $\|\cdot\|_\alpha$ is a cross-norm on the tensor product $A \otimes B$ such that the completion (denoted $A \otimes_\alpha B$) in this norm is a spectrally uniform Banach algebra. Then either both A and B are spectrally uniform, or one of them is uniformly topologically nil.*

Proof. With n fixed, for each $\epsilon > 0$ we can find $a \in A$ with

$$\|a^n\|^{1/n} - r(a) \geq (V_A(n) - \epsilon) \, \|a\|$$

and then for $b \in B$

$$\begin{aligned} \|(a \otimes b)^n\|_\alpha^{1/n} - r(a \otimes b) &= \|a^n\|^{1/n} \, \|b^n\|^{1/n} - r(a)r(b) \\ &\geq \left(\|a^n\|^{1/n} - r(a) \right) \|b^n\|^{1/n} \\ &\geq (V_A(n) - \epsilon) \, \|(a \otimes b)\|_\alpha \, \|b^n\|^{1/n} \, / \, \|b\| \, . \end{aligned}$$

So taking the supremum over $b \in B$ with $\|b\| \leq 1$ and letting $\epsilon \to 0$ we have

$$V_{A \otimes_\alpha B}(n) \geq \sup \left\{ \|b^n\|^{1/n} : b \in B, \, \|b\| \leq 1 \right\} V_A(n) \tag{2.5}$$

and it follows from (2.5) that if $A \otimes_\alpha B$ is spectrally uniform then either A is spectrally uniform or B is uniformly topologically nil. As this is true when A and B are interchanged the result follows. ∎

2.4. Dense Subalgebras

In this section we consider whether the properties of uniform convergence carry over from dense subalgebras of Banach algebras. We find such stability in the case of spectral uniformity, using a straightforward continuity-of-norm argument, but not in general for topologically bounded index.

Proposition 2.4.1. *Suppose that A is a Banach algebra and $B \subseteq A$ a cone. Then the equality $V_B(n) = V_{\overline{B}}(n)$ obtains for all $n \in \mathbb{N}$.*

Proof. For fixed n and assuming that $V_B(n) > 0$ we take $b \in \overline{B}$ and $b_k \in B$ with $b_k \to b$. For $\epsilon > 0$ we can find k_0 such that

$$\|b_k\| \leq (1 + \epsilon/2V_B(n)) \|b\| \quad (k \geq k_0)$$

and since the mapping $x \mapsto \|x^n\|^{1/n}$ is continuous in a Banach algebra we can find k_1 such that

$$\|b^n\|^{1/n} - \|b_k^n\|^{1/n} \leq \epsilon \|b\| / 4 \quad (k \geq k_1).$$

Similarly, in this case using the upper semi-continuity of the mapping $x \mapsto r(x)$ (see, for example, [8, Prop. 17, §5]), there is some k_2 with

$$r(b_k) \leq r(b) + \epsilon \|b\| / 4 \quad (k \geq k_2).$$

Hence

$$\|b^n\|^{1/n} - r(b) \leq \|b_k^n\|^{1/n} - r(b_k) + \epsilon \|b\| / 2 \quad (k \geq k_1,\, k_2)$$

and so for $k \geq k_0,\, k_1,\, k_2$

$$\begin{aligned} \frac{\|b^n\|^{1/n} - r(b)}{\|b\|} &\leq \frac{\|b_k^n\|^{1/n} - r(b_k)}{\|b_k\|} \left(1 + \frac{\epsilon}{2V_B(n)}\right) + \frac{\epsilon}{2} \\ &\leq V_B(n) + \epsilon. \end{aligned}$$

A simpler argument gives a similar estimate in the case that $V_B(n) = 0$ and letting $\epsilon \to 0$ then gives the result. □

Corollary 2.4.2. *A Banach algebra with a dense spectrally uniform subalgebra is spectrally uniform.*

Corollary 2.4.3. *A Banach algebra with a dense uniformly topologically nil subalgebra is uniformly topologically nil.*

Note that Corollary 2.4.3 shows that a non-radical Banach algebra with a dense subalgebra of topologically nilpotent elements cannot be of topologically bounded index. Examples of such algebras (in fact, semisimple examples) have been constructed by Dixon [17] and Hadwin *et al.* [24]. We will later consider the second of these in some detail but conclude this section by showing that for topologically bounded index the statement corresponding to the above corollaries can fail.

Recall that ℓ^1 is the Banach algebra of sequences $a = (\lambda_n)$ of complex numbers λ_n with

$$\|a\|_1 := \sum_{i=1}^{\infty} |\lambda_i| < \infty$$

and co-ordinate-wise algebraic operations.

If S is a semigroup we denote by $\mathbb{C}[S]$ the complex space of formal sums x given by

$$x = \sum_{s \in S} \lambda_s s$$

where only finitely many of the $\lambda_s \in \mathbb{C}$ are non-zero. With the natural summand-wise addition and scalar multiplication, and product

$$\left(\sum_{s \in S} \lambda_s s \right) \left(\sum_{s \in S} \mu_s s \right) = \sum_{s \in S} \left(\sum_{tu = s} \lambda_t \mu_u \right) s$$

$\mathbb{C}[S]$ is a complex algebra (the *semigroup algebra* of S). The completion of $\mathbb{C}[S]$ in the ℓ^1-norm

$$\left\| \sum_{s \in S} \lambda_s s \right\|_1 = \sum_{s \in S} |\lambda_s|$$

is a Banach algebra denoted $\ell^1(S)$. This algebra is considered in some detail in [43, Sect. 4.8.6].

Example 2.4.4. Let (e_i) denote the natural basis of ℓ^1 and for each m, $n \in \mathbb{N}$ write $L_{m,n}$ for the weighted left shift operator on ℓ^1 given by

$$L_{m,n} e_i = \begin{cases} 0 & \text{if } i = 1, \\ e_{i-1} & \text{if } i = 2, \dots, n+1, \\ (1/m) e_{i-1} & \text{if } i = n+2, \dots \end{cases}$$

Denote by FS_2 the free semigroup on symbols s, t and write $\ell^1(FS_2)$ for the ℓ^1 semigroup algebra of FS_2. We let $p_m = s^m t^m \in FS_2$ and define A_n to be the normed subalgebra of the Cartesian product $\ell^1(FS_2) \times \mathscr{B}(\ell^1)$ generated by the elements $(\frac{1}{m} p_m, L_{m,n})$ $(m \in \mathbb{N})$. The algebraic operations on A_n are co-ordinate-wise and the norm is the maximum of the norms of the two co-ordinates.

It is known that $\ell^1(FS_2)$ has no non-zero topologically nilpotent elements (see [8, Example 46.6]) and so if $(a, b) \in A_n$ is topologically nilpotent we have $a = 0$. Since A_n is generated by the $(\frac{1}{m} p_m, L_{m,n})$ there is some $M \in \mathbb{N}$ and a polynomial P such that

$$(a, b) = P\left((p_1, L_{1,n}), \dots (\tfrac{1}{M} p_M, L_{M,n}) \right)$$

so $P\left(p_1, \dots, \frac{1}{M} p_M\right) = 0$. But this implies that P is trivial since distinct products of the p_i produce distinct elements of FS_2. Hence A_n contains no non-zero topologically nilpotent elements.

Now let $L_n \in \mathscr{B}(\ell^1)$ be defined by

$$L_n e_i = \begin{cases} 0 & \text{if } i = 1, \\ e_{i-1} & \text{if } i = 2, \dots, n+1, \\ 0 & \text{if } i = n+2, \dots \end{cases}$$

so that

$$(L_{m,n} - L_n)e_i = \begin{cases} 0 & \text{if } i = 1, \dots, n+1, \\ (1/m)e_{i-1} & \text{if } i = n+2, \dots. \end{cases}$$

We then have, for $\sum_{i=1}^{\infty} \lambda_i e_i \in \ell^1$

$$\left\| (L_{m,n} - L_n) \sum_{i=1}^{\infty} \lambda_i e_i \right\|_1 = \frac{1}{m} \sum_{i=n+2}^{\infty} |\lambda_i| \leq \frac{1}{m} \left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|_1$$

so $\|L_{m,n} - L_n\| \leq 1/m$ for each $m \in \mathbb{N}$. It follows that $(0, L_n)$ is contained in the completion B_n of A_n since

$$\left\| (0, L_n) - \left(\tfrac{1}{m} p_m, L_{m,n}\right) \right\| \leq \frac{2}{m} \quad (m \in \mathbb{N})$$

and a short calculation shows that $\|(0, L_n)^n\| = \|(0, L_n)\| = 1$ while $(0, L_n)^{n+1} = 0$.

Our construction is almost complete. Let A denote the set of all bounded sequences whose n-th co-ordinate is in A_n. With co-ordinate-wise algebraic operations and supremum norm A becomes a normed algebra and using the above we see that A contains no non-zero topologically nilpotent elements. However, considering the sequence in B, the completion of A, with $(0, L_n)$ in the n-th co-ordinate and zero elsewhere, we see that $V_{T(B)}(n) = 1$ for each $n \in \mathbb{N}$.

2.5. Radical Extensions of Commutative Banach Algebras

In this section we ask if a commutative Banach algebra A is spectrally uniform given that both J-rad(A) and $A/\,J$-rad(A) are. We find a positive answer in many cases when A satisfies an extra structural condition.

A Banach algebra A possesses a *(strong) Wedderburn decomposition* if there is a closed subalgebra $B \subseteq A$ such that

$$A = B \oplus J\text{-rad}(A). \tag{2.6}$$

If such a decomposition exists then it is easy to see that $B \cong A/\,J\text{-rad}(A)$. Questions as to the existence and uniqueness of such decompositions have been addressed by numerous authors including Bade & Curtis [5, 6], Gorin & Lin [23] and more recently Albrecht & Ermert [1]. We will discuss some of the consequences of results obtained by these at the end of this section.

To begin, we fix some notation. Let A denote a commutative Banach algebra with a (not necessarily unique) Wedderburn decomposition as in (2.6). Then there is a constant C such that, for any $b \in B$, $r \in J$-rad(A) with $\|b + r\| \leq 1$, we have

$\|b\|, \|r\| \leq C$. This can be seen by applying the open mapping theorem to the mapping

$$\begin{array}{rcl} B \times \text{J-rad}(A) & \longrightarrow & A \\ (b, r) & \longmapsto & b + r \end{array}$$

where the Cartesian product has the max-norm. Throughout this section C will denote such a constant and we shall take $C > 1$ without loss of generality.

Theorem 2.5.1. *Suppose that A is a commutative Banach algebra with a (not necessarily unique) Wedderburn decomposition $A = B \oplus \text{J-rad}(A)$ and that B is spectrally uniform. Further suppose that f is a non-decreasing function $\mathbb{N} \to \mathbb{N}$ with*

$$f(n) = o\left(\frac{n}{\log n}\right)$$

and that

$$\lim_{n\to\infty} V_{\text{J-rad}(A)}\big(f(n)\big)^{f(n)/n} = 0.$$

Then A is spectrally uniform.

Proof. The conclusion of the theorem obviously holds if A has a trivial radical so we suppose henceforth that it does not.

Let $a \in A$ with $\|a\| \leq 1$ so that $a = b + r$ for unique $b \in B$, $r \in \text{J-rad}(A)$ and $\|b\|, \|r\| \leq C$. By the growth condition on f, along with the observation that $V_{\text{J-rad}(A)}(1) = 1$ (as the radical is non-trivial), we know that there is some n_0, depending only on f, such that

$$1 < f(n) < n/2 \quad (n \geq n_0).$$

We suppose that $n \geq n_0$ and consider the inequality

$$\|a^n\|^{1/n} \leq \left(\|b^n\| + \sum_{j=1}^{f(n)} \binom{n}{j} \|b^{n-j}\| \|r^j\|\right)^{1/n} + \left(\sum_{j=f(n)+1}^{n} \binom{n}{j} \|b^{n-j}\| \|r^j\|\right)^{1/n}. \tag{2.7}$$

For $n \geq n_0$ and $1 \leq j \leq f(n)$ we have $\binom{n}{j} \leq \binom{n}{f(n)}$, so

$$\begin{aligned} \|b^n\| + \sum_{j=1}^{f(n)} \binom{n}{j} \|b^{n-j}\| \|r^j\| &\leq \sum_{j=0}^{f(n)} \binom{n}{f(n)} \left\|b^{n-f(n)}\right\| C^{f(n)} \\ &\leq n^{f(n)+1} C^{f(n)} \left\|b^{n-f(n)}\right\| \\ &\leq (n^2 C)^{f(n)} \left\|b^{n-f(n)}\right\| \end{aligned}$$

and, since $r(a) = r(b)$,

$$\begin{aligned}
&\left(\|b^n\| + \sum_{j=1}^{f(n)} \binom{n}{j} \|b^{n-j}\| \|r^j\|\right)^{1/n} - r(a) \\
&\leq \left(Cn^2\right)^{f(n)/n} \left\|b^{n-f(n)}\right\|^{1/n} - r(b) \\
&\leq \left(Cn^2\right)^{f(n)/n} \big(C\, V_B\,(n - f(n)) + r(b)\big)^{(n-f(n))/n} - r(b). \qquad (2.8)
\end{aligned}$$

To obtain a bound on (2.8) we consider the functions g_n given by

$$g_n(t) = (Cn^2)^{f(n)/n} \left(C\, V_{J\text{-rad}(A)}(n - f(n)) + t\right)^{(n-f(n))/n} - t \quad (t \geq 0)$$

which have $g_n'(t) > 0$ whenever

$$t < Cn^2 \left(\frac{n - f(n)}{n}\right)^{n/f(n)} - C\, V_{J\text{-rad}(A)}(n - f(n)). \qquad (2.9)$$

Since the right-hand side of (2.9) tends to infinity with increasing n, there is some $n_1 \geq n_0$ such that for $n \geq n_1$ each g_n is increasing on $[0, C]$. Thus for $n \geq n_1$ the quantity (2.8) is no greater than

$$\begin{aligned}
&\left(Cn^2\right)^{f(n)/n} \big(C\, V_B\,(n - f(n)) + C\big)^{(n-f(n))/n} - C \\
&= C \left(n^{f(n)/n}\right)^2 \big(V_B\,(n - f(n)) + 1\big)^{(n-f(n))/n} - C \qquad (2.10)
\end{aligned}$$

and the growth condition on f shows that $n^{f(n)/n} \to 1$ so (2.10) tends to zero as n tends to infinity.

Thus, combining the inequalities (2.7), (2.8) and (2.10) we obtain

$$\begin{aligned}
\|a^n\|^{1/n} - r(a) \leq{}& C \left(n^{f(n)/n}\right)^2 \big(V_B\,(n - f(n)) + 1\big)^{(n-f(n))/n} - C \\
&+ \left(\sum_{j=f(n)+1}^{n} \binom{n}{j} \|b^{n-j}\| \|r^j\|\right)^{1/n} \qquad (2.11)
\end{aligned}$$

for $n \geq n_1$ and so we only need a bound on the final term of (2.11) to complete the proof. We have

$$\begin{aligned}
\sum_{j=f(n)+1}^{n} \binom{n}{j} \|b^{n-j}\| \|r^j\| &\leq \sum_{j=f(n)+1}^{n} \binom{n}{j} C^{n-j} \left\|r^{f(n)}\right\| C^{j-f(n)} \\
&\leq 2^n C^n \left\|r^{f(n)}\right\|
\end{aligned}$$

so that

$$\begin{aligned}\left(\sum_{j=f(n)+1}^{n}\binom{n}{j}\left\|b^{n-j}\right\|\left\|r^{j}\right\|\right)^{1/n} &\leq 2\,C\left\|r^{f(n)}\right\|^{1/n} \\ &\leq 2\,C\,V_{J\text{-rad}(A)}(f(n))^{f(n)/n}.\end{aligned}$$

Thus the required bound follows from the hypothesis on J-rad(A). □

The condition on J-rad(A) in Theorem 2.5.1 is rather technical so we include the following simple corollary.

Corollary 2.5.2. *Suppose that A is a commutative Banach algebra with a (not necessarily unique) Wedderburn decomposition $A = B \oplus J\text{-rad}(A)$, that B is spectrally uniform and that for some $\alpha > 0$*

$$V_{J\text{-rad}(A)}(n)^{1/n^{\alpha}} \to 0$$

as $n \to \infty$. Then A is spectrally uniform.

Proof. Take $k \in \mathbb{N}$ with $\alpha > 1/k$ and let $f(n)$ be the integer part of $n^{k/(k+1)}$. Then

$$\begin{aligned}V_{J\text{-rad}(A)}\Big(f(n^{k+1})\Big)^{f(n^{k+1})/n^{k+1}} &= V_{J\text{-rad}(A)}\Big(n^{k}\Big)^{1/n} \\ &\leq V_{J\text{-rad}(A)}\Big(n^{k}\Big)^{1/(n^{k})^{\alpha}} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Moreover

$$\frac{f(n)\log n}{n} \leq \frac{n^{k/(k+1)}\log n}{n} = \frac{\log n}{n^{1/(k+1)}} \to 0$$

as $n \to \infty$ so we may apply Theorem 2.5.1. □

Our next theorem is similar to the above in hypotheses, but differs in that we place restrictions on $A/\,J$-rad(A) rather than J-rad(A).

Theorem 2.5.3. *Suppose that A is a commutative Banach algebra with a (not necessarily unique) Wedderburn decomposition $A = B \oplus J\text{-rad}(A)$, such that B is isomorphic to a uniform algebra and J-rad(A) is uniformly topologically nil. Then A is spectrally uniform.*

Proof. We will write $\alpha_n := V_{\text{J-rad}(A)}(n)$ and

$$\beta_n := \sup\left\{\|b^n\|^{1/n}/r(b) :\ b \in B\backslash\{0\}\right\}.$$

Notice that these suprema exist because our assumption that B is isomorphic with a uniform algebra implies that there is a constant K with

$$r(b) \leq \|b\| \leq K r(b) \quad (b \in B)$$

and so $\beta_1 \leq K$. Then since

$$\|b^n\|^{1/n} \leq \beta_1^{1/n} r(b^n)^{1/n} = \beta_1^{1/n} r(b)$$

we have $\beta_n \leq \beta_1^{1/n}$, which also shows that $\beta_n \to 1$ as $n \to \infty$.

As before we take $a \in A$ with $\|a\| \leq 1$ so that $a = b + r$ for some $b \in B$, $r \in$ J-rad(A) with $\|b\|, \|r\| \leq C$. Note that if $r(a) = 0$ then, since A is commutative,

$$\|a^n\|^{1/n} - r(a) \leq V_{\text{J-rad}(A)}(n)$$

and so we need to find a bound on $\|a^n\|^{1/n} - r(a)$ supposing that $r(a) > 0$. Now

$$\begin{aligned}
\|a^n\| &\leq \|b^n\| + \sum_{k=1}^{n-1}\binom{n}{k}\|b^k\|\|r^{n-k}\| + \|r^n\| \\
&\leq \sum_{k=0}^{n}\binom{n}{k}\beta_k^k r(b)^k \alpha_{n-k}^{n-k}\|r\|^{n-k} \qquad (2.12)
\end{aligned}$$

and the isomorphism $A/\text{J-rad}(A) \cong B$ implies that $r(a) = r(b)$ so

$$\begin{aligned}
\|a^n\|^{1/n} - r(a) &\leq \left(\sum_{k=0}^{n}\binom{n}{k}\beta_k^k r(b)^k \alpha_{n-k}^{n-k}\|r\|^{n-k}\right)^{1/n} - r(b) \\
&= r(b)\left(\left(\sum_{k=0}^{n}\binom{n}{k}\beta_k^k \alpha_{n-k}^{n-k}\left(\frac{\|r\|}{r(b)}\right)^{n-k}\right)^{1/n} - 1\right). \qquad (2.13)
\end{aligned}$$

We now claim that for any $t > 0$

$$\left(\sum_{k=0}^{n}\binom{n}{k}\beta_k^k \alpha_{n-k}^{n-k} t^{n-k}\right)^{1/n} \to 1. \qquad (2.14)$$

To see this consider the complex algebra $\mathscr{A}_0$ generated by the commuting symbols $\mathbf{b}$ and $\mathbf{r}$. With the convention that $\mathbf{b}^0\mathbf{r}^i = \mathbf{r}^i$ and $\mathbf{b}^i\mathbf{r}^0 = \mathbf{b}^i$ for $i \geq 1$ we can write a typical element of $\mathscr{A}_0$ as

$$\sum \xi_{i,j}\mathbf{b}^i\mathbf{r}^j$$

where the summation is all pairs of non-negative integers (i, j) save $(0, 0)$, and only finitely many of the $\xi_{i,j} \in \mathbb{C}$ are non-zero. For such an element we define

$$\left\|\sum \xi_{i,j}\mathbf{b}^i\mathbf{r}^j\right\| := \sum \beta_i^i \alpha_j^j t^j |\xi_{i,j}|$$

which is a norm (a weighted ℓ^1 norm) on $\mathscr{A}_0$. To see that it is an algebra norm we note that

$$\begin{aligned}
\left\|\left(\mathbf{b}^i\mathbf{r}^j\right)\left(\mathbf{b}^k\mathbf{r}^l\right)\right\| &= \left\|\mathbf{b}^{i+k}\mathbf{r}^{j+l}\right\| \\
&= \beta_{i+k}^{i+k}\alpha_{j+l}^{j+l}t^{j+l} \\
&\leq \left(\beta_i^i\alpha_j^j t^j\right)\left(\beta_k^k\alpha_l^l t^l\right) \\
&= \left\|\mathbf{b}^i\mathbf{r}^j\right\|\left\|\mathbf{b}^k\mathbf{r}^l\right\|
\end{aligned}$$

which, by a routine argument, implies submultiplicativity. Thus $\mathscr{A}$, the completion of $\mathscr{A}_0$ in this norm, is a Banach algebra. One confirms that

$$J\text{-rad}(\mathscr{A}) = \overline{\operatorname{span}}\left\{\mathbf{b}^i\mathbf{r}^j : i \in \mathbb{N},\, j > 1\right\}$$

and so

$$\begin{aligned}
\left(\sum_{k=0}^{n}\binom{n}{k}\beta_k^k\alpha_{n-k}^{n-k}t^{n-k}\right)^{1/n} &= \|(\mathbf{b}+\mathbf{r})^n\|^{1/n} \\
&\to r(\mathbf{b}+\mathbf{r}) \\
&= r(\mathbf{b}) \\
&= \lim_{n\to\infty}\|\mathbf{b}^n\|^{1/n} = \lim_{n\to\infty}\beta_n = 1
\end{aligned}$$

which proves (2.14).

To complete the proof we take $\epsilon > 0$ and write β for the largest β_n. Let $n_0 \geq 2$ be such that $\alpha_n \leq \epsilon/4C$ for all $n \geq n_0$ and let $R = 12\beta C$.

We have, by (2.14) with $t = R/\epsilon$, that there is some $n_1 \in \mathbb{N}$ such that

$$\left(\sum_{k=0}^{n} \binom{n}{k} \beta_k^k \alpha_{n-k}^{n-k} \left(\frac{R}{\epsilon}\right)^{n-k}\right)^{1/n} - 1 \le \epsilon/C \quad (n \ge n_1)$$

and so by (2.13) if $\|r\| / r(b) \le R/\epsilon$

$$\|a^n\|^{1/n} - r(a) \le \epsilon \quad (n \ge n_1)\,. \tag{2.15}$$

In the case that $r(b)/\|r\| \le \epsilon/R$ and $n \ge n_0$ we have from (2.12)

$$\begin{aligned}
&\|a^n\|^{1/n} - r(a) \\
\le\;& \left(\sum_{k=0}^{n} \binom{n}{k} \beta_k^k r(b)^k \alpha_{n-k}^{n-k} \|r\|^{n-k}\right)^{1/n} \\
\le\;& C\left(\sum_{k=0}^{n} \binom{n}{k} \beta^k \alpha_{n-k}^{n-k} \left(\frac{r(b)}{\|r\|}\right)^{k}\right)^{1/n} \\
\le\;& C\left(\sum_{k=0}^{n} \binom{n}{k} \beta^k \alpha_{n-k}^{n-k} \left(\frac{\epsilon}{R}\right)^{k}\right)^{1/n} \\
\le\;& C\left(\sum_{k=0}^{n-n_0} \binom{n}{k} \beta^k \alpha_{n-k}^{n-k} \left(\frac{\epsilon}{R}\right)^{k}\right)^{1/n} + C\left(\sum_{k=n-n_0+1}^{n} \binom{n}{k} \beta^k \alpha_{n-k}^{n-k} \left(\frac{\epsilon}{R}\right)^{k}\right)^{1/n} \\
\le\;& C\left(\sum_{k=0}^{n-n_0} \binom{n}{k} \beta^k \left(\frac{\epsilon}{4C}\right)^{n-k} \left(\frac{\epsilon}{R}\right)^{k}\right)^{1/n} + C\left(\sum_{k=1}^{n} \binom{n}{k}\right)^{1/n} \beta \left(\frac{\epsilon}{R}\right)^{(n-n_0+1)/n} \\
\le\;& C\epsilon\left(\frac{\beta}{R} + \frac{1}{4C}\right) + 2C\beta\left(\frac{\epsilon}{R}\right)^{(n-n_0+1)/n} \\
\to\;& C\epsilon\left(\frac{\beta}{R} + \frac{1}{4C}\right) + 2C\beta\left(\frac{\epsilon}{R}\right) \\
=\;& \epsilon/2.
\end{aligned}$$

Thus there is $n_2 \in \mathbb{N}$ such that

$$\|a^n\|^{1/n} - r(a) \le \epsilon \quad (\|r\| / r(b) \ge R/\epsilon,\ n \ge n_2)$$

which, combined with (2.15), shows that A is spectrally uniform. □

It would be interesting to know if the hypotheses of this theorem can be weakened to assuming only that the β_n exist for sufficiently large n. The methods of our proof do not do not readily indicate how such a generalisation could be made.

We complete this section with some corollaries to the above theorems. In [6, Theorem 4.2], Bade & Curtis show that if A is a commutative Banach algebra with $A/\,J\text{-rad}(A) \cong C(X)$ for some compact, Hausdorff and totally disconnected space X, and if J-rad(A) is nil, then A possesses a (unique) Wedderburn decomposition. It was later shown, by Gorin & Lin in [23], that uniform topological nillity could replace nillity as a condition on J-rad(A) in Bade & Curtis's result. More recently Albrecht & Ermert [1] have shown that [6, Theorem 4.2] holds when X is not necessarily totally disconnected.

Corollary 2.5.4. *Suppose that A is a commutative Banach algebra with*

$$A/\,J\text{-rad}(A) \cong C(X)$$

for some compact Hausdorff space X and that

1. *J-rad(A) is nil, or*
2. *J-rad(A) is uniformly topologically nil and X is totally disconnected.*

Then A is spectrally uniform.

Proof. In either case the aforementioned results guarantee that A possesses a Wedderburn decomposition and since $A/\,J\text{-rad}(A) \cong C(X)$ the conditions of Theorem 2.5.3 are met. $\square$

A similar, but weaker, conclusion holds when $A/\,J\text{-rad}(A) \cong \ell^1$ again using results of Gorin & Lin. It is easy to see that ℓ^1 does not satisfy the hypothesis of Theorem 2.5.3. Indeed, since for any $a = (\alpha_n) \in \ell^1$ we have $r(a) = \max\{|\alpha_i| : i \in \mathbb{N}\}$ we can take a_k to be the sequence in ℓ^1 with ones in the first k co-ordinates and zero elsewhere, to find

$$\|a_k^n\|_1^{1/n}/r(a_k) = k^{1/n} \to \infty$$

as $k \to \infty$. Thus the β_n described in the proof of Theorem 2.5.3 do not exist for any n.

It is possible, however, to show that ℓ^1 is spectrally uniform. We will need the following lemma whose proof is based on Maclaurin's proof of the inequality of arithmetic and geometric means described in Hardy-Littlewood-Polya [25, 2.6(i)].

Lemma 2.5.5. *For $k \in \mathbb{N}$ let S_k denote the unit k-simplex*

$$S_k = \Big\{(x_1, \dots, x_k) \in \mathbb{R}^k : x_i \geq 0,\, x_1 + \cdots + x_k = 1\Big\}.$$

and define functions $g_{k,n} : S_k \to \mathbb{R}$ by

$$g_{k,n}(x_1, \dots, x_k) = (x_1^n + \cdots + x_k^n)^{1/n} - \max_{i=1,\dots,k} x_i.$$

Then

$$g_{k,n}(x_1,\ldots,x_k) \le 1/\left((n-2)e\right) \quad (n \ge 3,\, (x_1,\ldots,x_k) \in S_k)\,.$$

Proof. Let $(x_1,\ldots,x_k)$ be a point of S_k at which the continuous function $g_{k,n}$ attains its maximum. We suppose, without loss of generality, that

$$x_1 = x_2 = \cdots = x_p > x_{p+1},\ldots,x_k$$

and write x for the common value of $x_1,\ldots,x_p$.

First note that if $p = k$ then $x = 1/k$ and

$$g_{k,n}(1/k,\ldots,1/k) = (k^{1/n}-1)/k. \tag{2.16}$$

If $p < k$ then we set $\delta = \min\{|x - x_i| : i = p+1,\ldots,k\}$ and define a function f by

$$f(y) = g_{k,n}(x_1 - y,\ldots,x_p - y, x_{p+1},\ldots,x_{k-1}, x_k + py)$$

for $y \in [0,\delta]$. Then

$$f(y) = \left(p(x-y)^n + x_{p+1}^n + \cdots + x_{k-1}^n + (x_k+py)^n\right)^{1/n} - (x-y)$$

and f possess a continuous right derivative f'_+ on $[0,\delta)$ given by

$$\begin{aligned} f'_+(y) \;&=\; \left(-np(x-y)^{n-1} + np(x_k+py)^{n-1}\right) \times \\ &\quad \times (1/n)\left(p(x-y)^n + x_{p+1}^n + \cdots + x_{k-1}^n + (x_k+py)^n\right)^{1/n-1} + 1 \end{aligned}$$

so

$$f'_+(0) = 1 - p\left(x^{n-1} - x_k^{n-1}\right)\left(x_1^n + \cdots + x_k^n\right)^{1/n-1}. \tag{2.17}$$

By our assumption that $(x_1,\ldots,x_k)$ is a point at which $g_{k,n}$ attains its maximum, we must have that $f'_+(0) \le 0$. So by (2.17)

$$\begin{aligned} (x_1^n + \cdots + x_k^n)^{1/n} \;&\le\; p^{1/(n-1)}\left(x^{n-1} - x_k^{n-1}\right)^{1/(n-1)} \\ &\le\; p^{1/(n-1)} x \end{aligned}$$

and thus

$$\begin{aligned} g_{k,n}(x_1,\ldots,x_n) \;&=\; (x_1^n + \cdots + x_k^n)^{1/n} - x \\ &\le\; \left(p^{1/(n-1)} - 1\right) x \\ &\le\; \left(p^{1/(n-1)} - 1\right)/p. \end{aligned} \tag{2.18}$$

We obtain suitable bounds for both (2.16) and (2.18) as follows. For $n \ge 2$ let h_n be the real valued function on $(0,\infty)$ given by

$$h_n(\xi) = (\xi^{1/n} - 1)/\xi \quad (0 < \xi < \infty).$$

Apply the calculus to find that h_n attains its maximum when $\xi = (1-1/n)^{-n}$ and then

$$h_n(\xi) \leq \frac{1}{n-1}\left(1-\frac{1}{n}\right)^n \quad (0 < \xi < \infty).$$

Since $(1-1/n)^n < e^{-1}$ we have

$$h_n(\xi) \leq 1/\left((n-1)e\right) \quad (n \geq 2,\ 0 < \xi < \infty)$$

and so by (2.16) and (2.18)

$$\begin{aligned} g_{k,n}(x_1,\ldots,x_k) &\leq \begin{cases} 1/\left((n-1)e\right) & p = k \\ 1/\left((n-2)e\right) & p < k \end{cases} \\ &\leq 1/\left((n-2)e\right) \end{aligned}$$

for $n \geq 3$ as required. □

Proposition 2.5.6. *The Banach algebra ℓ^1 is spectrally uniform with*

$$V_{\ell^1}(n) \leq 1/n \quad (n \geq 4) \tag{2.19}$$

Proof. Suppose that $a \in \ell^1$ has finite support, say

$$a = (\alpha_1, \ldots, \alpha_k, 0, \ldots)$$

and that $\|a\|_1 = 1$. Then

$$\begin{aligned} \|a^n\|_1^{1/n} - r(a) &= \left(|\alpha_1|^n + \cdots + |\alpha_k|^n\right)^{1/n} - \max_{i=1,\ldots,k} |\alpha_i| \\ &\leq 1/\left((n-2)e\right) \end{aligned}$$

for $n \geq 3$ by the lemma. Thus, with F denoting the subalgebra of ℓ^1 of those sequences with finite support, we have

$$V_F(n) \leq 1/\left((n-2)e\right) \leq 1/n \quad (n \geq 4)$$

and this bound also holds for $V_{\ell^1}(n)$ by Proposition 2.4.1, which shows (2.19). □

Corollary 2.5.7. *Suppose that A is a commutative Banach algebra with*

$$A/\,\text{J-rad}(A) \cong \ell^1$$

and such that for some $\alpha > 0$

$$V_{\text{J-rad}(A)}(n)^{1/n^\alpha} \to 0$$

as $n \to \infty$. Then A is spectrally uniform.

Proof. The conditions that $A/\,$J-rad$(A) \cong \ell^1$ and that J-rad(A) is uniformly topologically nil guarantee that A has a Wedderburn decomposition by [23]. Now apply Corollary 2.5.2. □

3. Spectral Uniformity

3.1. Semisimple Commutative Banach Algebras

It is easy to see that a uniform algebra is spectrally uniform, since an element of such an algebra has its spectral radius equal to its norm. Moreover, any Banach algebra isomorphic to a uniform algebra is spectrally uniform by Proposition 2.1.1. Also it is known that a Banach algebra is isomorphic to a uniform algebra if, and only if, its spectral radius is equivalent to its norm (see [4, Th. 4.1.13]). Thus it is natural to ask if commutative Banach algebras in which the spectral radius is equivalent to the norm are the only commutative semisimple Banach algebras which are spectrally uniform. The following example gives a negative answer to this question.

Let $C^1[0,1]$ denote the space of continuously differentiable complex functions on $[0,1]$. With pointwise algebraic operations and norm

$$|||f||| := \|f\|_\infty + \|f'\|_\infty \quad \big(f \in C^1[0,1]\big)$$

$C^1[0,1]$ is a semisimple commutative Banach algebra with unit. Clearly $f \in C^1[0,1]$ is invertible if and only if $f(x) \neq 0 \; (x \in [0,1])$ and so

$$r(f) = \|f\|_\infty \quad \big(f \in C^1[0,1]\big)\,.$$

From this we see that the norm of $C^1[0,1]$ is not equivalent to its spectral radius.

Proposition 3.1.1. *The Banach algebra $C^1[0,1]$ is spectrally uniform with*

$$V_{C^1[0,1]}(n) < n^{1/n} - 1 \quad (n \geq 4).$$

Proof. Let $f \in C^1[0,1]$ with $|||f||| \leq 1$. Then for $n \geq 2$

$$\begin{aligned} |||f^n||| &= \|f^n\|_\infty + \|(f^n)'\|_\infty \\ &= \|f\|_\infty^n + n\|f'f^{n-1}\|_\infty \\ &\leq \|f\|_\infty^n + n\|f'\|_\infty \|f\|_\infty^{n-1} \\ &\leq n\|f\|_\infty^{n-1} \end{aligned}$$

so

$$|||f^n|||^{1/n} - r(f) \leq n^{1/n}\|f\|_\infty^{1-1/n} - \|f\|_\infty \quad \big(n \geq 2,\ f \in C^1[0,1],\ |||f||| \leq 1\big)\,. \tag{3.1}$$

Now consider the functions $F_n : [0, \infty) \to \mathbb{R}$ given by

$$F_n(x) = n^{1/n} x^{1-1/n} - x \quad (x \in [0, \infty)) ,$$

which are continuously differentiable on $(0, \infty)$ with

$$F_n'(x) = n^{1/n}(1 - 1/n)x^{-1/n} - 1 \quad (x \in (0, \infty)) .$$

Each F_n' is strictly decreasing with increasing x and zero when $x = n(1 - 1/n)^n$. Since $(1 - 1/n)^n$ is strictly increasing (to e^{-1}) and $(1 - 1/4)^4 > 1/4$ we have $n(1 - 1/n)^n > 1$ for all $n \geq 4$. Thus

$$F_n'(x) > 0 \quad (n \geq 4,\ x \in (0, 1))$$

and so F_n is non-decreasing on $[0, 1]$ for $n \geq 4$. Then, by (3.1)

$$\begin{aligned} |||f^n|||^{1/n} - r(f) &\leq F_n\left(\|f\|_\infty\right) \\ &\leq F_n(1) \\ &= n^{1/n} - 1 \quad (|||f||| \leq 1,\ n \geq 4) \end{aligned}$$

which provides the required bound on $V_{C^1[0,1]}(n)$. □

The bound on $V_{C^1[0,1]}(n)$ in the above proposition is dominated by $(e-1)\log(n)/n$, as can be seen by elementary arguments (see [25, Theorem 150], for example).

The example above suggests that we ask whether stronger conditions are sufficient to force a Banach algebra to be (isomorphic to) a uniform algebra. The next proposition shows that an abundance of n-th roots along with suitably fast uniform decay of $\|a^n\|^{1/n}/r(a)$ provide such conditions. First we need a simple consequence of the continuity of the map $a \mapsto a^n$ which is proved in [2].

Lemma 3.1.2. *Suppose that A is a Banach algebra with*

$$A^{[n]} := \{a^n : a \in A\}$$

dense in A. Then $A^{[n^2]}$, $A^{[n^4]}$, $A^{[n^8]}, \ldots$ are dense in A.

Proposition 3.1.3. *Suppose that A is a Banach algebra with no topologically nilpotent elements bar zero and that*

1. *there is some $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the numbers*

$$\beta_n := \sup\left\{\|a^n\|^{1/n}/r(a) : a \in A,\ a \neq 0\right\}$$

are defined,

2. *for some C we have $\beta_n \leq C^{1/n}$ for $n \geq n_0$, and*

3. for some $m \geq 2$, $A^{[m]}$ is dense in A.

Then A is isomorphic with a uniform algebra.

Proof. Suppose that $a \in A$ and that $k \in \mathbb{N}$ has $n = m^{2^k} > n_0$. Then given $\epsilon > 0$ we can find $b \in A$ with $\|b^n - a\|$ small enough to guarantee (by upper semicontinuity of the spectral radius) that

$$r(a) \geq r(b^n) - \epsilon,$$

and to force $\|b^n\| \geq (1-\epsilon)\|a\|$. Then

$$\begin{aligned} r(a) &\geq r(b)^n - \epsilon \\ &\geq \left(\|b^n\|^{1/n}/\beta_n\right)^n - \epsilon \\ &\geq (1-\epsilon)\|a\|/\beta_n^n - \epsilon \end{aligned}$$

so that, letting $\epsilon \to 0$,

$$\|a\| \leq r(a)\beta_n^n \leq C\,r(a) \quad (a \in A).$$

Thus A has its spectral radius equivalent to the norm, which means that A is isomorphic to a uniform algebra. $\square$

We remark that the abundance of roots described in the above is a restrictive condition. It is easy to see, for simply-connected compact Hausdorff X, that $C(X)$ satisfies this condition. But $C(\mathbb{T})$, the continuous functions on the unit circle, does not — consider a neighbourhood of the function $f \in C(\mathbb{T})$ with $f(z) = z$.

Notice, however, that the conditions on the spectral radius in above cannot be dropped. We have previously noted that ℓ^1 is spectrally uniform but does not satisfy condition 1 of the above proposition. But any sequence with finite support has a square root in ℓ^1 and such sequences form a dense subalgebra of ℓ^1.

We conclude with an example which shows that a commutative semisimple Banach algebra can fail to be spectrally uniform quite dramatically. We have already seen, in the examples of Section 2.2, that a commutative Banach algebra A may have $V_A(n) = 1$ ($n \in \mathbb{N}$). The following semisimple example also satisfies this condition.

Example 3.1.4. Let A_0 denote the complex algebra generated by the symbol a. Thus A_0 is the algebra of sums

$$x = \sum_{i=1}^{\infty} \lambda_i\, a^i$$

where only finitely many of the $\lambda_i \in \mathbb{C}$ are non-zero. Set

$$\omega_{m,n}(i) = \begin{cases} 1 & i = 1, 2, \ldots, n \\ 1/m^i & i = n+1, \ldots \end{cases}$$

and define norms $\|\cdot\|_{m,n}$ on A_0 by

$$\left\|\sum_{i=1}^{\infty} \lambda_i a^i\right\|_{m,n} = \sum_{i=1}^{\infty} |\lambda_i| \, \omega_{m,n}(i).$$

It is easy to see that

$$\omega_{m,n}(i+j) \leq \omega_{m,n}(i)\omega_{m,n}(i) \quad (i,\, j,\, m,\, n \in \mathbb{N})$$

and it follows that each $\|\cdot\|_{m,n}$ is an algebra norm. We write $A_{m,n}$ for the completion of A_0 in the norm $\|\cdot\|_{m,n}$.

Then

$$\|a\|_{m,n} = \left\|a^2\right\|_{m,n} = \cdots = \|a^n\|_{m,n} = 1$$

but

$$\left\|a^{n+1}\right\|_{m,n}^{1/(n+1)} = \left\|a^{n+2}\right\|_{m,n}^{1/(n+2)} = \cdots = 1/m$$

so that $V_{A_{m,n}}(n) \geq 1 - 1/m$ for $m,\, n \in \mathbb{N}$. We let A denote the Cartesian product of the $A_{m,n}$ and apply Proposition 2.3.2 to see that $V_A(n) = 1$ for all $n \in \mathbb{N}$.

To show that A is semisimple one applies essentially the same method as is used in Example 2.2.1.

3.2. Continuity of the Spectral Radius

Questions concerning the continuity of the spectral radius in a Banach algebra have been addressed by numerous authors. It is known that the spectral radius is always *upper-semicontinuous:* i.e. for any a in a Banach algebra A, and $\epsilon > 0$ there is $\delta > 0$ such that

$$r(b) \leq r(a) + \epsilon \quad (\|a - b\| < \delta). \tag{3.2}$$

For some Banach algebras more can be said. In the algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices the spectral radius is continuous (see [4, Th. 3.4.5] for Newburgh's proof of a more general fact). In a commutative Banach algebra A, the spectral radius is uniformly continuous on A (see [4, Th. 3.4.1]). If $B \subseteq A$ is a cone, then uniform continuity of the spectral radius on B is equivalent to the existence of a constant C such that

$$|r(a) - r(b)| \leq C \, \|a - b\| \quad (a,\, b \in B) \tag{3.3}$$

We find that the spectral radius in a spectrally uniform Banach algebra is uniformly continuous on the unit ball (which is not, of course, a cone) as follows.

Proposition 3.2.1. *Suppose that A is a spectrally uniform Banach algebra. Then there is a continuous function $F : [0,1] \to [0,3]$ with $F(t)$ decreasing monotonically to zero as t decreases to zero and such that*

$$|r(a) - r(b)| \leq F(\|a - b\|) \quad (a,\, b \in A,\ \|a\|\,,\, \|b\| \leq 1).$$

Proof. From the definition of spectral uniformity we know that

$$-r(b) \leq -\|b^n\|^{1/n} + V_A(n)\,\|b\| \quad (b \in A)$$

so that

$$r(a) - r(b) \leq \|a^n\|^{1/n} - \|b^n\|^{1/n} + V_A(n)\,\|b\| \quad (a,\, b \in A). \tag{3.4}$$

Supposing that $\|a\|\,,\, \|b\| \leq 1$ we take moduli in (3.4) to obtain the inequality

$$|r(a) - r(b)| \leq \left|\|a^n\|^{1/n} - \|b^n\|^{1/n}\right| + V_A(n) \quad (a,\, b \in A,\ \|a\|\,,\, \|b\| \leq 1). \tag{3.5}$$

Since we may write, for elements a, b of an algebra,

$$a^n - b^n = a^{n-1}(a-b) + a^{n-2}(a-b)b + \cdots + (a-b)b^{n-1},$$

we can quickly obtain

$$\begin{aligned} \left|\|a^n\|^{1/n} - \|b^n\|^{1/n}\right| &\leq \|a^n - b^n\|^{1/n} \\ &\leq n^{1/n}\,\|a-b\|^{1/n} \quad (a,\, b \in A,\ \|a\|\,,\, \|b\| \leq 1). \end{aligned}$$

Thus

$$|r(a) - r(b)| \leq n^{1/n}\,\|a-b\|^{1/n} + V_A(n) \quad (a,\, b \in A,\ \|a\|\,,\, \|b\| \leq 1)$$

and we write

$$F(t) = \inf_{n \in \mathbb{N}} \left\{ n^{1/n} t^{1/n} + V_A(n) \right\} \quad (t \in [0,1])$$

to obtain the required function. ☐

We remark that it is not true that a spectrally uniform Banach algebra necessarily has a uniformly continuous spectral radius as in (3.3). The spectral radius in the Banach algebra $M_n(\mathbb{C})$ is not uniformly continuous (see [4, §3.4]) but we shall see below that $M_n(\mathbb{C})$ is spectrally uniform.

3.3. Algebras of $k \times k$ Matrices

In this section we consider the spectral uniformity of the algebra $M_k(\mathbb{C})$ of complex $k \times k$ matrices. Fortunately much is known about the norms of powers of matrices since questions concerning them arise in problems whose solutions are given by iterative matrix schemes (see Young [55] and [53]). In particular, the following bound obtained by Young in [55] quickly leads to a proof that $M_k(\mathbb{C})$ is spectrally uniform.

Recall that the *operator norm* of an operator T on a Banach space $\mathscr{X}$ is the algebra norm given by

$$\|T\|_{\mathrm{op}} := \sup\{\|Tx\| : \|x\| \leq 1\}.$$

Theorem 3.3.1 (Young, 1981)**.** *For any $k \times k$ matrix T*

$$\|T^n\|_{\mathrm{op}} \leq \binom{n}{k-1} \|T\|_{\mathrm{op}}^{k-1} r(T)^{n-k+1} \quad (n \geq k) \tag{3.6}$$

where the norm $\|T\|_{\mathrm{op}}$ of T is that of T considered as an operator on k-dimensional Hilbert space.

Corollary 3.3.2. *The Banach algebra $M_k(\mathbb{C})$, with operator norm, is spectrally uniform with*

$$V_{M_k(\mathbb{C})}(n) \leq \binom{n}{k-1}^{1/n} - 1$$

for sufficiently large n.

Proof. We suppose that $T \in M_k(\mathbb{C})$ has $\|T\|_{\mathrm{op}} \leq 1$ and apply (3.6) to obtain

$$\|T^n\|_{\mathrm{op}}^{1/n} - r(T) \leq \binom{n}{k-1}^{1/n} r(T)^{1-(k-1)/n} - r(T) \quad (n \geq k). \tag{3.7}$$

As previously we define functions $f_n : (0, \infty) \to \mathbb{R}$ by

$$f_n(t) = \binom{n}{k-1}^{1/n} t^{1-(k-1)/n} - t$$

and apply the calculus to find that f_n' is strictly decreasing, and zero when

$$t = \binom{n}{k-1}^{1/(k-1)} \left(1 - \frac{k-1}{n}\right)^{n/(k-1)}.$$

Since this tends to infinity with n there is some n_0 such that f_n is strictly increasing on $(0,1]$ for $n \geq n_0$. Thus from (3.7) we have

$$\begin{aligned} \|T^n\|_{\mathrm{op}}^{1/n} - r(T) &\leq f_n\left(r(T)\right) \\ &\leq f_n(1) \\ &= \binom{n}{k-1}^{1/n} - 1 \quad (T \in M_k(\mathbb{C}),\ \|T\|_{\mathrm{op}} \leq 1,\ n \geq n_0) \end{aligned}$$

which provides the stated bound. □

To find a lower bound on $V_{M_k(\mathbb{C})}(n)$ we consider the $k \times k$ Jordan block matrix T; the matrix with ones on the diagonal and first superdiagonal, and zeros elsewhere. Note that since T is the sum of the identity and a shift it has norm no greater than 2. For $n \geq k-1$ we have

$$T^n = \begin{bmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{k-1} \\ & 1 & \binom{n}{1} & \cdots & \binom{n}{k-2} \\ & & 1 & \cdots & \binom{n}{k-3} \\ & 0 & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}$$

and if $x = (0, \ldots, 0, 1)^T$ then

$$\|T^n x\|^2 = 1 + \binom{n}{1}^2 + \cdots + \binom{n}{k-1}^2 > \binom{n}{k-1}^2 \quad (n \geq k-1).$$

Thus $\|T^n\|_{\mathrm{op}} > \binom{n}{k-1}$ and so

$$V_{M_k(\mathbb{C})}(n) \geq \frac{\|T^n\|_{\mathrm{op}}^{1/n} - r(T)}{\|T\|_{\mathrm{op}}} > \frac{1}{2}\left(\binom{n}{k-1}^{1/n} - 1\right) \quad (n \geq k-1). \tag{3.8}$$

The above matrix T has convergence to its spectral radius of order $\log(n)/n$, and this seems to be an unusual property. The author organised a computational competition of around 40 of the matrices in the MATLAB test matrix toolbox (described in [30]). The matrix T has by far the slowest convergence, at least to the 30-th power of a 10×10 matrix. Some matrices, similar in structure to T, have the norms of their powers illustrated in Figure A.4 of the appendix.

We can obtain neater upper and lower bounds on $V_{M_k(\mathbb{C})}(n)$ using the following straightforward inequalities. The proofs are included for completeness.

Proposition 3.3.3. *For $n \geq k \geq 2$*

$$\binom{n}{k-1}^{1/n} - 1 \geq \frac{\log n}{n}$$

and for $n \geq \log(2k-2)$, $k \geq 2$

$$\binom{n}{k-1}^{1/n} - 1 \leq (k-1)(e-1)\frac{\log n}{n}.$$

Proof. Since

$$\binom{n}{k-1} = \frac{n \cdots (n-k+2)}{(k-1)\dots 2} = n\left(\frac{n-1}{k-1}\right)\dots\left(\frac{n-k+2}{2}\right)$$

we have $\binom{n}{k-1} \geq n$ for $n \geq k \geq 2$. Writing $n = e^x$ we have

$$\binom{n}{k-1}^{1/n} - 1 \geq n^{1/n} - 1 = \exp\left(x\, e^{-x}\right) - 1 \quad (n \geq k \geq 2)$$

and since $e^y - 1 \geq y$ ($y \in \mathbb{R}$) this shows that

$$\binom{n}{k-1}^{1/n} - 1 \geq x\, e^{-x} = \frac{\log n}{n} \quad (n \geq k \geq 2).$$

To obtain the second inequality we first note that $\binom{n}{k-1} \leq n^{k-1}$. Again with $n = e^x$ we find that

$$\binom{n}{k-1}^{1/n} - 1 \leq n^{(k-1)/n} - 1 = \exp\left((k-1)x\, e^{-x}\right) - 1. \tag{3.9}$$

Now, by Taylor,

$$\begin{aligned} \frac{e^x}{(k-1)x} &= \frac{1}{(k-1)x} + \frac{1}{k-1} + \frac{x}{2k-2} + \cdots \\ &> \frac{1}{k-1} + \frac{x}{2k-2} \quad (x > 0) \end{aligned}$$

so that $e^x/((k-1)x) > 1$ whenever $x \geq 2k-2$ and thus

$$(k-1)x\, e^{-x} < 1 \quad (n > \log(2k-2)).$$

So since

$$e^y - 1 \leq (e-1)y \quad (0 \leq y \leq 1)$$

we have that

$$n^{(k-1)/n} - 1 \leq (k-1)(e-1)\frac{\log n}{n} \quad (n \geq \log(2k-2)) \tag{3.10}$$

and so, from (3.9),

$$\binom{n}{k-1}^{1/n} - 1 \leq (k-1)(e-1)\frac{\log n}{n} \quad (n \geq \log(2k-2))$$

as required. □

Corollary 3.3.4. *The algebra $M_k(\mathbb{C})$, with the operator norm, satisfies*

$$\frac{1}{2}\frac{\log n}{n} \leq V_{M_k(\mathbb{C})}(n) \leq (k-1)(e-1)\frac{\log n}{n}$$

for all sufficiently large n.

We now consider algebra norms other than the operator norm. Let A denote $M_k(\mathbb{C})$ equipped with some algebra norm $\|\cdot\|$. Of course $\|\cdot\|$ is equivalent to $\|\cdot\|_{\mathrm{op}}$ since $M_k(\mathbb{C})$ is semisimple [8, Th. 9, §25] and so A is spectrally uniform by Proposition 2.1.1. We can show a little more: that $V_A(n)$ is also asymptotically $\log(n)/n$.

Proposition 3.3.5. *Let A denote $M_k(\mathbb{C})$ equipped with some algebra norm $\|\cdot\|$ and let C be a positive constant such that*

$$C^{-1}\|T\| \leq \|T\|_{\mathrm{op}} \leq C\|T\| \quad (T \in A).$$

Then for all sufficiently large n

$$\frac{1}{4C}\,\frac{\log n}{n} \leq V_A(n) \leq 2C(k-1)(e-1)\,\frac{\log n}{n}.$$

Proof. For non-zero T in A we have

$$\begin{aligned}
\frac{\|T^n\|^{1/n} - r(T)}{\|T\|} &\leq \frac{C^{1/n}\|T^n\|_{\mathrm{op}}^{1/n} - r(T)}{C^{-1}\|T\|_{\mathrm{op}}} \\
&= C\left(\left(C^{1/n}-1\right)\frac{\|T^n\|_{\mathrm{op}}^{1/n}}{\|T\|_{\mathrm{op}}} + \frac{\|T^n\|_{\mathrm{op}}^{1/n} - r(T)}{\|T\|_{\mathrm{op}}}\right) \\
&\leq C\left(C^{1/n} - 1 + \binom{n}{k-1}^{1/n} - 1\right) \quad (n \geq k-1) \qquad (3.11)
\end{aligned}$$

by Corollary 3.3.2. Since $\binom{n}{k-1} \leq n^{k-1}$ we have from (3.11) that

$$\begin{aligned}
\frac{\|T^n\|^{1/n} - r(T)}{\|T\|} &\leq 2C\left(n^{(k-1)/n} - 1\right) \\
&\leq 2C(k-1)\,(e-1)\,\frac{\log n}{n} \qquad \left(n \geq C^{1/(k-1)},\, n \geq k-1\right)
\end{aligned}$$

using the equation (3.10) of Proposition 3.3.3 while noting that $k-1 > \log(2k-2)$ for $k > 1$.

To show the other inequality we proceed similarly. We let T denote the $k \times k$ Jordan block matrix. Then for $n \geq k$

$$\begin{aligned}
\frac{\|T^n\|^{1/n} - r(T)}{\|T\|} &\geq \frac{C^{-1/n}\,\|T^n\|_{\mathrm{op}}^{1/n} - r(T)}{C\,\|T\|_{\mathrm{op}}} \\
&= C^{-1}\left(\left(C^{-1/n} - 1\right)\frac{\|T^n\|_{\mathrm{op}}^{1/n}}{\|T\|_{\mathrm{op}}} + \frac{\|T^n\|_{\mathrm{op}}^{1/n} - r(T)}{\|T\|_{\mathrm{op}}}\right) \\
&\geq C^{-1}\left(C^{-1/n} - 1 + \frac{1}{2}\left(\binom{n}{k-1}^{1/n} - 1\right)\right) \qquad (3.12)
\end{aligned}$$

by (3.8). We have previously noted that $\binom{n}{k-1} \geq n$ for $n \geq k$ so writing $n = e^x$ we have

$$\begin{aligned}
\frac{1}{2}\left(\binom{n}{k-1}^{1/n} - 1\right) &\geq \frac{1}{2}\left(n^{1/n} - 1\right) \\
&= \frac{1}{2}\left(\exp\left(x\,e^{-x}\right) - 1\right) \\
&\geq \frac{1}{2}x\,e^{-x}
\end{aligned}$$

since $e^y - 1 \geq y \;\; (y \in \mathbb{R})$. Similarly, writing $C = e^K$,

$$C^{-1/n} - 1 = \exp\left(-K\,e^{-x}\right) - 1 \geq -K\,e^{-x}$$

so that by (3.12)

$$\begin{aligned}
\frac{\|T^n\|^{1/n} - r(T)}{\|T\|} &\geq C^{-1}\left(\frac{1}{2}x\,e^{-x} - K\,e^{-x}\right) \quad (n \geq k) \\
&= \frac{1}{nC}\left(\frac{1}{2}\log n - K\right) \quad (n \geq k) \\
&\geq \frac{1}{4C}\frac{\log n}{n} \quad \left(n \geq k,\, n \geq C^4\right).
\end{aligned}$$

□

We conclude this section with a slight extension to Corollary 3.3.2. Recall that an element a of an algebra A is *algebraic of degree* k if there is some polynomial p, of degree k, such that $p(a) = 0$. The Cayley-Hamilton Theorem tells us that each $k \times k$ matrix is algebraic of degree k.

Theorem 3.3.6 (Young [54]). *Suppose that A is a Banach algebra and that $a \in A$ with $\|a\| \leq 1$ is algebraic of degree k. Then*

$$\|a^n\| \leq \sum_{j=0}^{k-1}\binom{n}{j}(-1)^{n-j-1}\left(1 + r(a)\right)^j r(a)^{n-j} \quad (n \geq k). \qquad (3.13)$$

Corollary 3.3.7. *Suppose that A is a Banach algebra and that $a \in A$ with $\|a\| \le 1$ is algebraic of degree k. Then*

$$\|a^n\|^{1/n} - r(a) \le \left(2^{k-1}k\binom{n}{k-1}\right)^{1/n} - 1$$

for all sufficiently large n.

Proof. From (3.13) we have

$$\begin{aligned}
\|a^n\| &\le \sum_{j=0}^{k-1}\binom{n}{j}2^j r(a)^{n-j} \\
&\le 2^{k-1}r(a)^{n-k+1}\sum_{j=0}^{k-1}\binom{n}{j} \\
&\le 2^{k-1}r(a)^{n-k+1}k\binom{n}{k-1} \quad (n \ge 2k-2)
\end{aligned}$$

and so

$$\|a^n\|^{1/n} - r(a) \le \left(2^{k-1}r(a)^{n-k+1}k\binom{n}{k-1}\right)^{1/n} - r(a) \quad (n \ge 2k-2).$$

As previously, write

$$f_n(t) = \left(2^{k-1}k\binom{n}{k-1}\right)^{1/n} t^{1-(k-1)/n} - t$$

for $t > 0$. This defines a function, continuously differentiable on $(0, \infty)$, with

$$f_n'(t) = \left(1 - \frac{k-1}{n}\right)\left(2^{k-1}k\binom{n}{k-1}\right)^{1/n} t^{(1-k)/n} - 1.$$

We have $f_n'(t) > 0$ whenever

$$t < \left(1 - \frac{k-1}{n}\right)^{n/(k-1)} 2k^{1/(k-1)}\binom{n}{k-1}^{1/(k-1)}$$

and since this tends to infinity as $n \to \infty$ there is n_0 such that f_n' is positive on $(0,1)$ for $n \ge n_0$. Hence

$$\|a^n\|^{1/n} - r(a) \le \left(2^{k-1}k\binom{n}{k-1}\right)^{1/n} - 1 \quad (n \ge n_0) \tag{3.14}$$

as required. ☐

As in the matrix case we can make the bound (3.14) a little tidier by noting that

$$\begin{aligned} 2^{k-1}k\binom{n}{k-1} &\leq n\binom{n}{k-1} \\ &\leq n^k \quad (n \geq 2^{k-1}k) \end{aligned}$$

and so

$$\left(2^{k-1}k\binom{n}{k-1}\right)^{1/n} - 1 \leq n^{k/n} - 1 \quad (n \geq n_0 \geq 2,\, n \geq 2^{k-1}k)).$$

Now using equation (3.10) in Proposition 3.3.3 we find that

$$\left(2^{k-1}k\binom{n}{k-1}\right)^{1/n} - 1 \leq k(e-1)\frac{\log n}{n}$$

for all sufficiently large n.

Corollary 3.3.8. *Suppose that A is a Banach algebra each element of which is algebraic of degree k. Then*

$$V_A(n) \leq k(e-1)\frac{\log n}{n}$$

for all sufficiently large n.

Banach algebras satisfying the hypotheses of Corollary 3.3.8 are a rather restricted class. It is known that such algebras are exactly those which are finite modulo a nilpotent radical, as is shown by Dixon in [13].

3.4. Von Neumann Algebras

We here consider some subalgebras of the Banach algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on a Hilbert space $\mathscr{H}$. Recall that a *von Neumann algebra* is a *-subalgebra of $\mathscr{B}(\mathscr{H})$, closed in the *weak operator topology* — the locally convex topology induced by the family of seminorms $\rho_{x,y}$

$$\rho_{x,y}(T) = |\langle Tx, y\rangle| \quad (T \in \mathscr{B}(\mathscr{H}),\, x,\, y \in \mathscr{H})$$

A von Neumann algebra is always a C^**-algebra:* a *-subalgebra of $\mathscr{B}(\mathscr{H})$ closed in the norm topology. Von Neumann algebras have a rich structure theory which enables us to characterize spectral uniformity of such algebras in terms of their *type decomposition.* We refer the reader to [47, Chapter 10] for a detailed treatment as we need only the following facts.

1. A von Neumann algebra A admits a decomposition

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_\infty \oplus A_C$$

into orthogonal von Neumann subalgebras $A_1,\ A_2,\ \ldots,\ A_\infty$ and A_C where A_i is of type I_i (or trivial) for $i = 1, \ldots, \infty$ and A_C is continuous (or trivial) [47, 4.17, E4.14 & E4.15].

2. A type I_k von Neumann algebra is isometrically isomorphic to the algebra $C(X, M_k(\mathbb{C}))$ of continuous functions from some compact Hausdorff space X, to the $k \times k$ matrices over $\mathbb{C}$, with pointwise product and supremum norm [34, 6.6.5].

3. If A is a continuous or type I_∞ von Neumann algebra then for each $k \in \mathbb{N}$ there are projections $e_1, \ldots, e_k \in A$ with $e_1 + \cdots + e_k = 1$ and which are equivalent and pairwise orthogonal. [47, 4.12] & [34, 6.5.6].

Proposition 3.4.1. *A von Neumann algebra A is spectrally uniform if and only if its type decomposition is is the direct sum*

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_m \tag{3.15}$$

where A_k is type I_k (or trivial) for $k = 1, \ldots, m$.

Proof. We first note that, by Proposition 2.3.1, if A is the direct sum of finitely many orthogonal closed subalgebras then it is spectrally uniform if, and only if, all its summands are. Thus, to show that a von Neumann algebra A with a type decomposition as in (3.15) is spectrally uniform, it suffices to show that a type I_k von Neumann algebra is spectrally uniform.

If B is a type I_k von Neumann algebra then we identify B with the algebra $C(X, M_k(\mathbb{C}))$ as mentioned above and suppose that $f : X \to M_k(\mathbb{C})$. Then we have $f(x) \in M_n(\mathbb{C})$ and so

$$\begin{aligned}
\|f^n\|^{1/n} &= \left(\sup_{x\in X} \|f^n(x)\|_{\mathrm{op}}\right)^{1/n} \\
&= \sup_{x\in X} \|(f(x))^n\|_{\mathrm{op}}^{1/n} \\
&\leq \sup_{x\in X} \Big(\|f(x)\|_{\mathrm{op}} V_{M_k(C)}(n) + r\left(f(x)\right)\Big) \\
&\leq \|f\| \, V_{M_k(C)}(n) + \sup_{x\in X} r\left(f(x)\right) \\
&\leq \|f\| \, V_{M_k(C)}(n) + r(f)
\end{aligned}$$

since $\sigma\left(f(x)\right) \subseteq \sigma(f)$ $(x \in X)$ by a straightforward calculation. Hence a type I_k von Neumann algebra is spectrally uniform. This shows that a von Neumann algebra A with a type decomposition as in (3.15) is spectrally uniform. Moreover for some $C > 0$ and all sufficiently large n the inequality

$$\begin{aligned} V_A(n) &\leq C\left(\max_{k=1,\ldots,m} V_{M_k(\mathbb{C})}(n) + m^{1/n} - 1\right) \\ &\leq C\left(\binom{n}{m-1}^{1/n} - 1 + m^{1/n} - 1\right) \end{aligned}$$

obtains. This can be seen by combining equation (2.4) of Proposition 2.3.1 with the bound for $V_{M_k(\mathbb{C})}(n)$ given in Corollary 3.3.2. Using the same methods as in Proposition 3.3.5 one can quickly see that $V_A(n)$ is $O\left(\log(n)/n\right)$, as with matrix algebras.

To see the reverse implication we first note that each type I_k algebra contains an element v with

$$\|v^{k-1}\| = \|v\| = 1, \quad v^k = 0 \tag{3.16}$$

(consider the function whose constant value is the matrix with ones on the first superdiagonal and zeros elsewhere). This shows that a von Neumann algebra whose type decomposition contains type $\mathrm{I}_{k(i)}$ summands for an increasing sequence $k(i)$ is not even of topologically bounded index.

The remaining case is when the type decomposition of A contains continuous or type I_∞ summands. Since we may write $A = (A_1 \oplus \cdots) \oplus A_\infty \oplus A_C$ we see that it suffices to show that a continuous or type I_∞ algebra B is not spectrally uniform. To see this apply the third fact listed above to B to obtain, for each k, projections $e_1, e_2, \ldots, e_k$ which are orthogonal, equivalent and with sum 1. For $i = 1, 2, \ldots, k-1$ denote by $v_{i,i+1}$ a partial isometry implementing the equivalence $e_{i+1} \sim e_i$. Then for $1 \leq i < j \leq k$ define

$$v_{i,j} = v_{i,i+1} v_{i+1,i+2} \cdots v_{j-1,j}.$$

One quickly confirms, using a routine Hilbert space orthogonality argument, that $v = v_{1,2} + v_{2,3} + \cdots + v_{k-1,k}$ satisfies the condition (3.16) and so B is not even of topologically bounded index. $\square$

A von Neumann algebra with a type decomposition as in (3.15) is called *a finite sum of type* I_k *algebras* by Johnson in [33, Sect. 6].

Note that the proof of Proposition 3.4.1 shows a little more than its statement — that the following are equivalent for a von Neumann algebra A:

1 A is spectrally uniform,

2 A is of topologically bounded index and

3 A is a finite sum of type I_k algebras.

In fact there is co-incidence with some other well-known finiteness properties. A Banach algebra A is *subhomogeneous* if there is some $N \in \mathbb{N}$ such that all continuous irreducible representations of A are of dimension no greater than N, and A *satisfies a polynomial identity* if there is some polynomial $p(X_1, \ldots, X_n)$, in noncommuting indeterminates $X_1, \ldots, X_n$, with

$$p(a_1, \ldots, a_n) = 0 \quad (a_1, \ldots, a_n \in A).$$

In [33, Prop. 6.1] Johnson shows that for a C^*-algebra A both of these conditions are equivalent to the weak-operator closure of A in $\mathscr{B}(\mathscr{H})$, $\overline{A}^{\mathrm{w}}$ (which is a von Neumann algebra) being a finite sum of type I_k algebras. So for von Neumann algebras A we can add

4 A is subhomogeneous, and

5 A satisfies a polynomial identity

to our list of equivalent conditions.

Finally, and perhaps most interestingly, a recent result of Aristov shows that we can add

6 A is injective in the sense of Varopoulos

to this list. A Banach algebra is *injective in the sense of Varopoulos* (or just *injective*) if the product mapping from the injective tensor product

$$\begin{array}{rcl} A \otimes_\epsilon A & \longrightarrow & A \\ a \otimes b & \longmapsto & ab \end{array}$$

is bounded. In [3] Aristov shows that a C^*-algebra is subhomogeneous if and only if it is injective. We warn the reader that there is a clash of notation in the literature, with 'injective' possessing a quite different meaning in the context of von Neumann algebras. In the sequal we shall always mean 'injective in the the sense of Varopoulos' when we write 'injective', even in the case of von Neumann algebras.

These equivalences beg the question as to whether they hold for a wider class than just von Neumann algebras. This question is addressed in the final chapter.

4. Topologically Bounded Index[1]

4.1. Introduction

We have previously mentioned that our investigation of Banach algebra of topologically bounded index is motivated by two main considerations.

Firstly any Banach algebra A which is spectrally uniform is of topologically bounded index. The weaker condition, however, often proves to be more tractable — in particular when we have sufficient information on $T(A)$, the set of topologically nilpotent elements. For example, we obtain some information on which semigroups S have $\ell^1(S)$ of topologically bounded index. The corresponding questions for spectral uniformity seem much more difficult. We do not even know if $\ell^1(\mathbb{Z})$, the Wiener algebra, is spectrally uniform.[2]

Secondly there is the question of analogy with rings of bounded index. It seems that much of the work of the ring-theorists does not give useful information in the context of Banach algebras — the consequences of bounded index for a ring are often true for *all* Banach algebras. An exception to this is a theorem of Jacobson on the invertibility of elements with left inverses (Theorem 4.3.1) which leads quickly to a topological analogue providing helpful information for several classes of Banach algebras.

We begin with some examples showing that, in general, bounded index and topologically bounded index are quite different properties.

Example 4.1.1. The commutative Banach algebra $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{C}$, with supremum norm and convolution product

$$f * g(s) := \int_0^s f(t)g(s-t)dt \quad (f,\, g \in C[0,1])$$

is known to be uniformly topologically nil (this is a straightforward induction argument). For $n = 2, 3, \ldots$ the functions

$$f_n(t) = \begin{cases} 0 & 0 \le t \le 1/n \\ \dfrac{nt-1}{n-1} & 1/n < t \le 1 \end{cases} \tag{4.1}$$

[1] An abridged version of this chapter is to appear as 'Banach algebras of topologically bounded index' in the Bulletin of the Australian Mathematical Society.

[2] In 2003 Anders Dahlner (Lund University) provided me with an elegant proof that $\ell^1(\mathbb{Z})$ is not spectrally uniform

in $C[0,1]$ have $f_n^n = 0$. By Titchmarsh's theorem [48, Th. VII] we have that for any continuous function g on $[0,1]$, $g^m = 0$ implies that $g(t)$ is zero on $[0,1/p)$ for some $p < m$. Thus $f_n^m \neq 0$ for $m < n$ and so $C[0,1]$ contains nilpotent elements of arbitrarily large index: it is not of bounded index.

Example 4.1.2. Let ω be the weight function $\omega : [0,\infty) \to (0,\infty)$, given by

$$\omega(t) = \exp(-te^{-t}) \quad (t \in [0,\infty))\,.$$

Then the algebra $L^1(\omega)$ of Lebesgue measurable functions $f : [0,\infty) \to \mathbb{C}$, with convolution product and weighted L^1 norm

$$\|f\|_\omega := \int_0^\infty |f(t)|\omega(t)dt \quad (f \in L^1(\omega))$$

is a commutative radical Banach algebra. However $L^1(\omega)$ has a bounded approximate identity which, by [37, Prop. 2.4] and [18, Th. 2.1], is incompatible with it being uniformly topologically nil. Thus $L^1(\omega)$ is not of topologically bounded index, but an application of Titchmarsh's theorem shows that it has no non-zero nilpotent elements and so is vacuously of bounded index.

The differences between bounded index and topologically bounded index are illustrated diagramatically in the appendix.

4.2. Relationship with the Radical

As we have previously mentioned, the Jacobson radical J-rad(A) of a Banach algebra A can be characterized as the largest ideal in $T(A)$ (see [8, Theorem 25.1]) and if A is a commutative Banach algebra we have J-rad$(A) = T(A)$. Thus a commutative Banach algebra is of topologically bounded index if and only if its radical is uniformly topologically nil. In fact this is true for a more general class of Banach algebras: those which satisfy a polynomial modulo the radical (i.e. A/J-rad(A) satisfies a polynomial identity). To show this we make use of the generalised Gelfand transform. We sketch the theory here but refer the reader to [36, Chapter VI], or the survey article [39], for the details.

Suppose that A is a unital Banach algebra which satisfies a polynomial identity modulo the radical. Then there exists $n \in \mathbb{N}$ such that $a \in A$ is invertible if and only if $\pi(a)$ is invertible for all representations $\pi : A \to M_n(\mathbb{C})$ with the property that

$$|\pi(a)_{i,j}| \leq \|a\| \quad (a \in A,\, 1 \leq i,j \leq n). \tag{4.2}$$

With this value of n fixed, the set of representations satisfying (4.2), which we denote Φ_A, can be given a compact Hausdorff topology. If $C(\Phi_A, M_n(\mathbb{C}))$ denotes the Banach algebra of continuous functions from Φ_A to $M_n(\mathbb{C})$ then *the generalised Gelfand transform* is the linear mapping

$$\begin{array}{ccc} A & \longrightarrow & C(\Phi_A, M_n(\mathbb{C})) \\ a & \longmapsto & \hat{a} \end{array}$$

where $\hat{a}(\pi) = \pi(a)$ ($a \in A$, $\pi \in \Phi_A$). It is known that:

1. the generalised Gelfand transform is a continuous homomorphism,
2. $a \in A$ is in the radical if and only if $\hat{a} = 0$, and
3. the spectrum of $a \in A$ is the union of the spectra of $\hat{a}(\pi)$ ($\pi \in \Phi_A$).

In order to prove our proposition for general, rather than just unital, Banach algebras we will need the following observation. Recall that the *unitization* A^+ of a Banach algebra A is the Banach algebra of pairs (a, λ) ($a \in A$, $\lambda \in \mathbb{C}$) with co-ordinate-wise addition and scalar product, product

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu) \quad (a, b \in A,\ \lambda, \mu \in \mathbb{C})$$

and norm $\|(a, \lambda)\| = \|a\| + |\lambda|$.

Lemma 4.2.1. *A Banach algebra A is of topologically bounded index if, and only if, its unitization is of topologically bounded index.*

Proof. Note that if $(a, \lambda) \in T(A^+)$ then

$$\|(a, \lambda)^n\| = \|(a^n + \cdots, \lambda^n)\| = \|a^n + \cdots\| + |\lambda^n| \geq |\lambda|^n$$

and so $\lambda = 0$. The result now follows since the mapping $a \mapsto (a, 0)$ is an isometric monomorphism. $\square$

Proposition 4.2.2. *Let A be a Banach algebra which satisfies a polynomial identity modulo the radical. Then A is topologically bounded index if and only if J-rad(A) is uniformly topologically nil.*

Proof. That a topologically bounded index Banach algebra has a uniformly topologically nil radical is obvious. Moreover a unitization argument, using the lemma above, shows that we need only prove the converse in the case when A is unital.

So suppose that J-rad(A) is uniformly topologically nil and $a \in A$ is topologically nilpotent. Using the above notation we have

$$\sigma(a) = \bigcup \{\sigma(\hat{a}(\pi)) : \pi \in \Phi_A\} = \{0\}\,.$$

Hence the matrix $\hat{a}(\pi)$ is topologically nilpotent and so, by Cayley-Hamilton, $(\hat{a}(\pi))^n = 0$ $(\pi \in \Phi_A)$. Consequently $\widehat{(a^n)} = 0$ and so $a^n \in J\text{-rad}(A)$. Thus whenever $a \in T(A)$ with $\|a\| \leq 1$ we find

$$\left\|a^{kn}\right\|^{1/kn} = \left\|(a^n)^k\right\|^{1/kn} \leq V_{J\text{-rad}(A)}(k)^{1/n} \quad (k \in \mathbb{N})$$

and taking the supremum over all such a we obtain

$$V_{T(A)}(kn) \leq V_{J\text{-rad}(A)}(k)^{1/n} \quad (k \in \mathbb{N}).$$

The proposition now follows from Lemma 1.6. $\square$

Corollary 4.2.3. *A semisimple Banach algebra satisfying a polynomial identity is of topologically bounded index.*

The corollary contrasts with Example 3.1.4 which is a semisimple commutative Banach algebra as far from spectral uniformity as is possible.

It is easy to find semisimple Banach algebras which are not of of topologically bounded index. By Proposition 3.4.1, and the subsequent discussion, any von Neumann algebra which is not the finite sum of type I_k algebras provides such an example.

4.3. A Topological Jacobson Theorem

Many of the results on rings of bounded index do not seem to have analogues in the topological case. The following theorem of Jacobson [32] is a welcome exception.

Theorem 4.3.1 (Jacobson, 1950)**.** *If A is a ring (with unit) of bounded index then for any $a, b \in A$ with $ab = 1$ we have $ba = 1$.*

The topological version has a weaker (and topological) conclusion.

Theorem 4.3.2. *Let A be a unital normed algebra of topologically bounded index and suppose there are $a, b \in A$ with $ab = 1$. Then either*

1. *$ba = 1$ or*
2. *$\|b^n a^n\|$ is not bounded.*

Proof. We proceed as in the proof of Jacobson's theorem. Supposing that $ab = 1 \neq ba$ we define matrix units by

$$e_{i,j} = b^{i-1}(1 - ba)a^{j-1} \quad (i, j \in \mathbb{N}).$$

It is quickly confirmed that

$$e_{i,j}e_{k,l} = \delta_j^k e_{i,l} \quad (i, j, k, l \in \mathbb{N})$$

and that the $e_{i,j}$ are linearly independent.

Now write, for each $n \in \mathbb{N}$

$$v_n = \sum_{i=1}^{n} e_{i,i+1}$$

so that

$$v_n^n b^n = 1 - ba \neq 0, \quad v_n^{n+1} = 0 \quad (n \in \mathbb{N})$$

(which proves Jacobson's theorem since it shows that A contains nilpotents of arbitrarily large index so it is not of bounded index). For our topological version, we assume in addition that $\|b^n a^n\|$ is bounded. Since $v_n^n b^n = 1 - ba$ we have

$$\|1 - ba\| = \|v_n^n b^n\| \leq \|v_n^n\| \, \|b^n\| \quad (n \in \mathbb{N})$$

and so

$$\|v_n^n\|^{1/n} \geq \frac{\|1 - ba\|^{1/n}}{\|b^n\|^{1/n}} \geq \frac{\|1 - ba\|^{1/n}}{\|b\|} \quad (n \in \mathbb{N}). \tag{4.3}$$

Thus, for sufficiently large n,

$$\|v_n^n\|^{1/n} \geq \frac{1}{2\,\|b\|}. \tag{4.4}$$

Now, by hypothesis, there is some $K > 0$ such that

$$\|b^n a^n\| \leq K \quad (n \in \mathbb{N})$$

and since

$$v_n = \sum_{i=1}^{n} e_{i,i+1} = (1 - b^n a^n) a \quad (n \in \mathbb{N})$$

we have

$$\|v_n\| \leq (1 + \|b^n a^n\|) \, \|a\| \leq (1 + K) \, \|a\| \quad (n \in \mathbb{N}). \tag{4.5}$$

Combining (4.4) and (4.5) we find that, for sufficiently large n,

$$V_{T(A)}(n) \geq \frac{\|v_n^n\|^{1/n}}{\|v_n\|} \geq \frac{1}{2\,\|a\|\,\|b\|\,(1 + K)}$$

and so A is not of topologically bounded index. $\square$

At first sight it seems that the above theorem can be strengthened in the following fashion. If we replace the second statement in the conclusions of Theorem 4.3.2 by

2′. $\|b^n a^n\| \, \|a^n\|^{1/n}$ is not bounded.

then a proof of the corresponding theorem can proceed just as above, except that one discards the second inequality of (4.3). Note, however, that since we assume $ab = 1$ we have

$$\|a^n\|^{1/n} \|b^n\|^{1/n} \geq \|a^n b^n\|^{1/n} = 1$$

for all n, so $r(a)r(b) \geq 1$. In particular, neither a nor b is topologically nilpotent and it follows that 2 and 2′ are equivalent statements in this context.

It would be interesting know if there is a unital Banach algebra A of topologically bounded index with $a, b \in A$ and $ab = 1 \neq ba$. In other words: does the second possibility in the conclusions of Theorem 4.3.2 ever actually happen? One naturally thinks of algebras of operators on some Banach space, in particular of algebras containing shift operators, for such an example. However we see later (in Proposition 4.5.2) that this approach is frustrated by the fact that most such algebras are not of topologically bounded index.

Our attempts to construct a 'generators and relations' Banach algebra, which would answer this question have also foundered, but for reasons of algebraic complexity and difficulty in describing the topologically nilpotent elements of such a construction.

Finally we remark that algebras (and rings) satisfying the conclusions of Jacobson's Theorem have been the subject of some study (in, for example, [9], [26] and [38]). Such algebras are called *von Neumann finite* due to the fact that von Neumann algebras satisfying this condition are exactly the finite von Neumann algebras. (Montgomery [38] mentions a discussion of this fact in Dixmier [12, Ch. 3, §4], or see [22] for an excellent historical perspective.) However a type II_1 von Neumann algebra is finite [47, 4.21] but is neither of bounded index nor of topologically bounded index. Thus no converse is possible for Jacobson's Theorem or its topological version.

4.4. The ℓ^1-algebra of a Semigroup

The problem addressed in this section is the description of those semigroups S for which $\ell^1(S)$ is of topologically bounded index. We are able to find some sufficient, and some necessary conditions on S but find that these are not exhaustive.

To find a sufficient condition we can use the fact that a semisimple Banach algebra satisfying a polynomial identity is of topologically bounded index (Corollary 4.2.3). It is known that if S is a group then $\ell^1(S)$ is a semisimple Banach algebra. In fact, this is also true if S is merely an *inverse semigroup:* a semigroup S such that for each $s \in S$ there is a unique $t \in S$ satisfying $sts = s$ and $tst = t$. This fact is shown by Barnes in [7].

Now, when S is a group, the algebra $\mathbb{C}[S]$ satisfies a polynomial identity if, and only if, S is abelian-by-finite (see [42, Th. 3, Ch. 18] for example). Since the satisfaction of a polynomial identity extends to the closure we obtain that $\ell^1(S)$ is of topologically bounded index whenever S is an abelian-by-finite group. In [41, Th. 4], Okniński even establishes criteria for an inverse semigroup S to have $\mathbb{C}[S]$ satisfying a polynimial identity. Since these criteria are somewhat technical, and not used subsequently, we will omit their description.

To describe necessary conditions on S for $\ell^1(S)$ to be of topologically bounded index, we will need some definitions. A *periodic* element $s \in S$ is one with $\langle s \rangle := \{s, s^2, \ldots\}$ finite. For such an element there are unique $m = m(s)$, $k = k(s) \in \mathbb{N}$ with s, s^2, $\ldots$, s^{m+k-1} distinct, $s^{m+k} = s^m$ and $\langle s \rangle = \{s, s^2, \ldots, s^{m+k-1}\}$ (see [10]). In this case we say that s has *index* m and *period index* k, a situation described diagramatically in Figure 4.1. A semigroup consisting entirely of periodic elements is, naturally, said to be periodic.

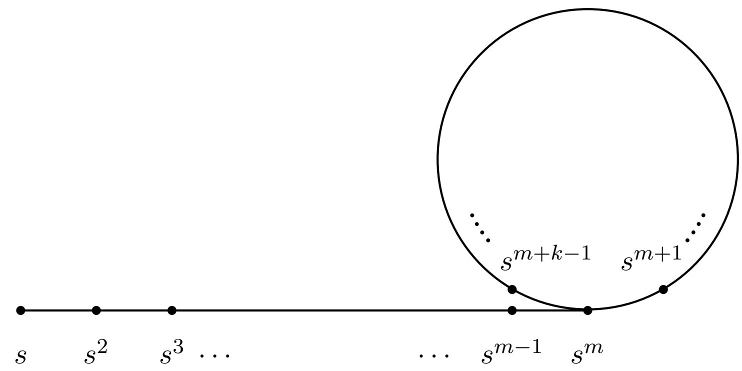

Figure 4.1.: A Periodic Element s of a Semigroup

Theorem 4.4.1. *If S is a semigroup such that $\ell^1(S)$ is topologically bounded index then the set*

$$\{m(s)/k(s) : s \in S \text{ is periodic}\} \tag{4.6}$$

is bounded or empty.

Proof. We will use the notation $[t]$ for the integer part of $t \in \mathbb{R}$. The result will follow from the fact that if some $s \in S$ is periodic with index m and period k then $V_{T(\ell^1(S))}([m/k]) = 1$ provided $2k \leq m$. For such s we write

$$x = \frac{1}{2}(s - s^k) \in \ell^1(S)$$

so that $\|x\|_1 = 1$ and if we write $d = [m/k]$ we have

$$\left\|x^d\right\|_1 = \frac{1}{2^d}\left\|\sum_{r=0}^{d}(-1)^r\binom{d}{r}s^{(d-r)+rk}\right\|_1 = \frac{1}{2^d}\sum_{r=0}^{d}\binom{d}{r} = 1$$

since $s^d, s^{d-1+k}, \ldots, s^{dk}$ are distinct by choice of d. Moreover we find that

$$x^m = \frac{1}{2^m}\sum_{r=0}^{m}(-1)^r\binom{m}{r}s^{(m-r)+rk} = 0$$

since $s^{m-r+rk} = s^{m+(k-1)r} = \cdots = s^m$ for $r = 0, \ldots, m$. □

As we can see from the following example, the set in (4.6) may be non-empty and may fail to be bounded away from zero.

We will need the following well-known characterization due to Hewitt and Zuckerman [29]; if S is a commutative semigroup then the condition that $s = t$ whenever $s, t \in S$ satisfy $s^2 = t^2 = st$, is equivalent to the condition that $\ell^1(S)$ is semisimple.

Example 4.4.2. Let $G_n = \left\{g_n,\, g_n^2, \ldots,\, g_n^{n-1},\, e_n\right\}$ denote the cyclic group of order n with unit e_n. Write $S = \cup_{n\in\mathbb{N}}G_n$ and define multiplication in S by

$$g_n g_m = g_{\max\{n,m\}}$$

so that S is a commutative semigroup.

Now let $s,\, t \in S$, say $s = g_n^k$, $t = g_m^k$, and suppose that $s^2 = t^2 = st$. Then obviously $m = n$, and so $s^2 = st$ implies that $g_n^{2k} = g_n^{k+l}$. Since g_n is an an element of the group G_n we have then that

$$s = g_n^k = g_n^l = g_m^l = t,$$

and so the aforementioned result of Hewitt and Zuckerman shows that $\ell^1(S)$ is semisimple. Thus $\ell^1(S)$ is of topologically bounded index, but each g_n is an element of S with index 1 and period n.

To show that Theorem 4.4.1 does not characterize semigroups S with $\ell^1(S)$ of topologically bounded index, we make use of the topological Jacobson theorem.

Example 4.4.3. Let B denote the *bicyclic semigroup:* the semigroup generated by symbols a, b and 1, subject to the relations $ab = 1$. Thus, with the convention that the symbols a^0 and b^0 both denote 1,

$$B = \{b^n a^m :\, n,\, m = 0, 1, \ldots\}.$$

When a,b and 1 are considered as elements of the algebra $\ell^1(B)$ (of course 1 is then the unit of $\ell^1(B)$, so the notation is consistent) we have $ab = 1 \neq ba$.

Moreover $\|b^n a^n\|_1 = 1$ for all $n \in \mathbb{N}$, in particular, $\|b^n a^n\|_1$ is bounded. Hence, by the topological Jacobson theorem (Theorem 4.3.2), $\ell^1(B)$ is not of topologically bounded index.

An element of B is, with the above convention, of the form $b^n a^m$ for some non-negative integers m, n. If $m = n$ then this element is idempotent and so has index and period 1. If, however, $n > m$ we find that

$$(b^n b^m)^k = b^{n+k(n-m)} a^m \quad (k \in \mathbb{N})$$

by an easy induction argument and thus $b^n a^m$ is not periodic. A symmetric argument shows that this is also true when $n < m$, and so the set in (4.4.1) is just the singleton $\{1\}$.

The observation that $\ell^1(B)$ is is not of topologically bounded index also leads to a curious structural constraint for certain semigroups.

A *zero* of a semigroup S is a (necessarily unique) element θ satisfying $s\theta = \theta s = \theta\,(s \in S)$. Note that if θ is considered as an element of $\ell^1(S)$ then it is *not* the zero element — hence the notation. A semigroup S with zero is *0-simple* if $S^2 \neq \{\theta\}$, and the only ideals of S are S and $\{\theta\}$.

An idempotent in a semigroup S with zero is said to be *primitive* if it is non-zero and minimal with respect to the partial ordering $\leq$ on the idempotents of S given by

$$e \leq f \Leftrightarrow ef = fe = e \quad (e, f \in S \text{ are idempotents}).$$

If S is a semigroup with zero then S is *completely 0-simple* if it is 0-simple and contains a primitive idempotent. These semigroups are of interest as they have an explicit structure theory, developed by D. Rees in the 1940s. Since this theory is developed in all textbooks on semigroup theory (see [31, Ch. III,§2 & 3], for example), we omit a full description here. Loosely speaking such a semigroup is isomorphic to a semigroup of certain matrices with entries from a group with zero adjoined.

The following theorem is from the doctoral thesis of O. Anderson (1952): a proof may be found in [10, Th. 2.34].

Theorem 4.4.4 (Anderson)**.** *Suppose that S is a 0-simple but not completely 0-simple semigroup, and that S contains an idempotent. Then S contains a sub-semigroup isomorphic to the bicyclic semigroup.*

Corollary 4.4.5. *Suppose that S is a 0-simple semigroup containing a non-zero idempotent and such that $\ell^1(S)$ is of topologically bounded index. Then S is completely 0-simple.*

Proof. If S is not completely 0-simple then, by Anderson's theorem, S has a subsemigroup isomorphic to the bicyclic semigroup B. But then $\ell^1(S)$ contains a subalgebra isomorphic to $\ell^1(B)$, which we have seen is not of topologically bounded index. ☐

We remark that Okniński has shown in [41] that the same conclusions hold under the hypotheses that S is a 0-simple semigroup containing a non-zero idempotent and such that $\mathbb{C}[S]$ satisfies a polynomial identity.

4.5. Algebras of Operators on a Banach Space

For an infinite-dimensional Hilbert space $\mathscr{H}$, the algebra $\mathscr{B}(\mathscr{H})$ of bounded operators on $\mathscr{H}$, is not of topologically bounded index (it is a type I_∞ von Neumann algebra). So for a Banach space $\mathscr{X}$, it would seem natural to suppose that $\mathscr{B}(\mathscr{X})$ is not of topologically bounded index. The following argument shows that this is true for many Banach spaces although we have not been able show full generality. We start with an old result due to Banach, a proof of which may be found in [52, Ch. II, Sect. B, Prop. 6].

Recall that a *Schauder basis* for a Banach space $\mathscr{X}$ is a sequence (e_i) of elements of $\mathscr{X}$, such that for each $x \in \mathscr{X}$ there is a unique sequence (λ_i) of complex numbers with

$$\sum_{i=1}^{\infty} \lambda_i e_i = x.$$

Theorem 4.5.1 (Banach 1932). *Let $\mathscr{X}$ be a Banach space with a Schauder basis (e_i). Then the projections*

$$P_n : \sum_{i=1}^{\infty} \lambda_i e_i \longmapsto \sum_{i=1}^{n} \lambda_i e_i$$

are bounded and $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$.

Proposition 4.5.2. *Suppose that $\mathscr{X}$ is a Banach space possessing a Schauder basis (e_i) and that the left and right unicellular shift operators, denoted L, R and defined by $L(e_1) = 0$, $L(e_n) = e_{n-1}\,(n = 2, 3, \ldots)$ and $R(e_n) = e_{n+1}\,(n \in \mathbb{N})$, relative to this basis are bounded. Then any subalgebra of $\mathscr{B}(\mathscr{X})$ which contains L and R is not of topologically bounded index.*

Proof. Suppose the contrary and note that $LR = 1$ but $RLe_1 = 0$ so $RL \neq 1$. Then by the topological Jacobson theorem we have that $\|R^n L^n\|$ is not bounded. But since $R^n L^n = 1 - P_n$ we have

$$\|R^n L^n\| \leq 1 + \|P_n\|$$

and so $\|P_n\|$ is not bounded, contrary to the above theorem of Banach. □

We now discuss a property stronger than topologically bounded index, but which seems intuitively close to it. Instead of asking if $\|a^n\|^{1/n}$ converges uniformly to zero over unit-norm topologically nilpotent a, we consider the uniform convergence of $\|a_1 \dots a_n\|^{1/n}$. We restrict our attention to $a_1, \dots, a_n$ in some semigroup of topologically nilpotent elements, so as to produce a property stronger than topologically bounded index. So consider the condition, on a Banach algebra A, that

$$\sup \left\{ \left(\frac{\|a_1 \dots a_n\|}{\|a_1\| \dots \|a_n\|} \right)^{1/n} : a_1, \dots, a_n \in S,\ S \subseteq T(A) \text{ is a semigroup} \right\} \to 0 \tag{4.7}$$

as $n \to \infty$. This may be considered to be to topologically bounded index what topological nilpotence (as in [18]) is to uniformly topological nillity.

It is fairly easy to see that (4.7) implies topologically bounded index. Indeed we know of only one example (due to Dixon and Müller in [19]) of a Banach algebra which is topologically bounded index but does not satisfy (4.7). Also (4.7) coincides with topological nilpotence for radical Banach algebras, and with topologically bounded index for commutative Banach algebras. The following suggests that it is better behaved.

Proposition 4.5.3. *If $\mathscr{X}$ is an infinite-dimensional Banach space then $\mathscr{B}(\mathscr{X})$ does not satisfy (4.7).*

Proof. Let n be fixed and take $\mathscr{Y}$ to be an $n+1$-dimensional subspace of $\mathscr{X}$. By a result of Auerbach (proved in [52, Ch. II, Sect. E, Lemma. 11]) we can find linearly independent vectors $e_1, \dots, e_{n+1} \in \mathscr{Y}$ and $f_1, \dots, f_{n+1} \in \mathscr{Y}^*$ all of unit norm and such that

$$f_i(e_j) = \delta_{i,j} \quad (i, j = 1, \dots, n+1).$$

Using the Hahn-Banach theorem we may extend each f_i to a linear functional (also denoted f_i) on $\mathscr{X}$ with the same (i.e. unit) norm. Now define, for $i = 1, \dots, n$

$$\begin{array}{rcl} T_i : \mathscr{X} & \longrightarrow & \mathscr{X} \\ x & \longmapsto & f_i(x) e_{i+1}. \end{array}$$

Note that

$$\|T_i x\| = |f_i(x)|\, \|e_{i+1}\| \le \|f_i\|\, \|x\| = \|x\|$$

so that $\|T_i\| \le 1$ and that

$$T_i T_j x = f_i(f_j e_{j+1}) e_{i+1} = f_i(e_{j+1}) f_j(x) e_{i+1}$$

so $T_i T_j = 0$ for $i \neq j+1$. Thus the semigroup S generated by $\{T_1, \dots, T_n\}$ consists of nilpotent operators. We have that $T_1, \dots, T_n \in S \subseteq T(\mathscr{B}(\mathscr{X}))$ and

$$T_n \cdots T_1 e_1 = e_{n+1}$$

so that $\|T_n \cdots T_1\| = 1$, which completes the proof. ∎

4.6. The Algebra of Hadwin et al. and a Related Matrix

In the note [24] Hadwin *et al.* construct a nil algebra $\mathscr{A}$, of operators on a separable Hilbert space, whose norm closure is semisimple. Using a non-constructive and operator-theoretic argument they show that this algebra is not of bounded index, a fact which also follows from the observation that bounded index extends to the closure. Since topologically bounded index also extends to the closure (Corollary 2.4.3), $\mathscr{A}$ is not of topologically bounded index.

It is the object of this section to show that

$$V_{T(\mathscr{A})}(2^n - 1) = 1 \quad (n \in \mathbb{N})$$

and, in so doing, to analyse a matrix (in fact a sequence of matrices) which may be of independent interest.

We first describe the construction of the algebra of Hadwin *et al.* Let $\mathscr{H}$ be the separable infinite dimensional Hilbert space with orthonormal basis (e_i). Given an $n \times n$ matrix $T = [t_{i,j}]$, considered as an operator acting on the first n basis elements of $\mathscr{H}$, define the operator $\mathrm{amp}\,(T) \in \mathscr{B}(\mathscr{H})$ by

$$\mathrm{amp}\,(T)e_{nk+i} = t_{1,i}e_{nk+1} + \cdots + t_{n,i}e_{nk+n} \quad (k = 0, 1, \ldots,\ i = 1, \ldots, n-1).$$

In other words, $\mathrm{amp}\,(T)$ acts on $\mathscr{H}$ as the infinite matrix

$$\begin{bmatrix} T & & \\ & T & \\ & & \ddots \end{bmatrix}.$$

In the above matrix, and subsequently, we denote zeros by blank spaces. It is not difficult to see that, for any matrix T, we have $\|\mathrm{amp}\,(T)\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}$.

For any $n \times n$ matrix T we shall write $N(T)$ for the $2n \times 2n$ matrix

$$\begin{bmatrix} T & -T \\ T & -T \end{bmatrix}$$

and make the slight abuse of exponential notation by writing

$$N^k(T) = N\underbrace{\Big(N\big(\cdots N\big(N(T)\big)\cdots\big)\Big)}_{k \text{ parentheses}} \tag{4.8}$$

so that for an $n \times n$ matrix T, $N^k(T)$ is a $2^k n \times 2^k n$ matrix. Note that for any matrix T we have $\|N(T)\|_{\mathrm{op}} = 2\,\|T\|_{\mathrm{op}}$.

For $n = 0, 1, \ldots$, write

$$\mathscr{A}_n = \{\mathrm{amp}\,(N(T)) :\ T \in M_{2^n}(\mathbb{C})\} \subseteq \mathscr{B}(\mathscr{H})$$

and
$$\mathscr{A} = \bigcup_{k=0}^{\infty} \sum_{n=0}^{k} \mathscr{A}_n.$$

This is the algebra constructed by Hadwin *et al.* In [24] it is shown that the norm closure $\overline{\mathscr{A}}$ of $\mathscr{A}$ in $\mathscr{B}(\mathscr{H})$ is semisimple, and it is observed that $\mathscr{A}$ is nil as follows. We have
$$\mathscr{A}_n \mathscr{A}_m \subseteq \begin{cases} \mathscr{A}_{\max\{n,m\}} & n \neq m \\ \{0\} & n = m \end{cases} \qquad (n, m = 0, 1, \ldots),$$

since $N(S)N(T) = 0$ for matrices S, T of equal dimension. Since each $T \in \mathscr{A}$ can be written, for some n, as $T = T_0 + \cdots + T_{n-1}$ with $T_i \in \mathscr{A}_i$ we have
$$\begin{aligned} T^2 &\in \mathscr{A}_1 + \mathscr{A}_2 + \cdots + \mathscr{A}_{n-1} \\ T^4 &\in \mathscr{A}_2 + \cdots + \mathscr{A}_{n-1} \\ &\vdots \\ T^{2^{n-1}} &\in \mathscr{A}_{n-1} \end{aligned}$$

and so $T^{2^n} = 0$.

Theorem 4.6.1. *For each $n \in \mathbb{N}$ there is a matrix $B_n \in M_{2^n}(\mathbb{C})$ satisfying*
$$\begin{gathered} \operatorname{amp}(B_n) \in \mathscr{A}_0 + \mathscr{A}_1 + \cdots + \mathscr{A}_{n-1} \\ \left\| B_n^{2^n - 1} \right\|_{\mathrm{op}} = \| B_n \|_{\mathrm{op}} = 1, \end{gathered}$$

and consequently $V_{T(\mathscr{A})}(2^n - 1) = 1$ for $n \in \mathbb{N}$.

The construction of the matrices B_n is straightforward, but demonstrating that they satisfy the conclusions of the theorem is not. The proof, therefore, is broken into a series of lemmas.

To construct the B_n we must introduce some new notation. For $S \in M_n(\mathbb{C})$ let
$$P(S) = \begin{bmatrix} S & S \\ -S & -S \end{bmatrix} \quad \left(= N\left(S^T\right)^T \right) \in M_{2n}(\mathbb{C})$$

where S^T, as usual, is the transpose of S. We will use the notational device P^k as with N^k in equation (4.8).

For each $n \in \mathbb{N}$ let
$$W_n = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in M_n(\mathbb{C})$$

so that W_nT is, for $T \in M_n(\mathbb{C})$, the matrix with the rows of T in reversed order. Similarly TW_n is the matrix with the columns of T reversed. We will only consider W_n when multiplied by a matrix of a specified order, so there is no ambiguity in writing WT for W_nT. This we do for the remainder of this section.

We will use a finite dimensional version of the amplification operator $\operatorname{amp}(T)$ as follows. For $0 \le m \le n$ we write

$$\begin{aligned} \operatorname{amp}_m^n : M_{2^m}(\mathbb{C}) &\longrightarrow M_{2^n}(\mathbb{C}) \\ T &\longmapsto \begin{bmatrix} T & & & \\ & T & & \\ & & \ddots & \\ & & & T \end{bmatrix}. \end{aligned}$$

We can now define the matrices B_n as follows: set $A_0 = 1/2$ and

$$A_k = \frac{1}{2} P\,(WA_{k-1}) \quad (k \in \mathbb{N})$$

so that $A_k \in M_{2^k}(\mathbb{C})$ for each k. Then let

$$B_n = \sum_{k=0}^{n-1} \operatorname{amp}_{k+1}^n (N(A_k)) \in M_{2^n(\mathbb{C})} \quad (n \in \mathbb{N}).$$

The recursive definition of the matrices B_n makes them particulary easy to calculate using a computer package such as MATLAB. The first three[3] are

$$B_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad B_2 = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -3 \\ -1 & -1 & 3 & -1 \end{bmatrix}$$

$$B_3 = \frac{1}{8}\begin{bmatrix} 5 & -3 & -3 & -3 & 1 & 1 & 1 & 1 \\ 3 & -5 & -3 & 3 & -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & -5 & -1 & -1 & -1 & -1 \\ -3 & -3 & 5 & -3 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -7 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 \\ -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \end{bmatrix}.$$

It is obvious that $\operatorname{amp}(B_n) \in \mathscr{A}_0 + \cdots + \mathscr{A}_{n-1}$ and so to prove the Theorem we need to calculate some of the norms mentioned therein.

[3] B_3 corrected, 18 June 08

Lemma 4.6.2. *The matrices $X_n \in M_{2^n}(\mathbb{C})$, defined inductively by*

$$X_1 = N(A_0)$$
$$X_k = \operatorname{amp}_{k-1}^{k}(X_{k-1}) N(A_{k-1}) \operatorname{amp}_{k-1}^{k}(X_{k-1}) \quad (k = 2, 3, \ldots), \tag{4.9}$$

satisfy $B_n^{2^n-1} = X_n$ for $n \in \mathbb{N}$.

Proof. We write $m = 2^n - 1$ to simplify our notation. First note that the definition of the B_n implies that B_n^m is the sum of the products

$$\operatorname{amp}_{k_1+1}^{n}(N(A_{k_1})) \ldots \operatorname{amp}_{k_m+1}^{n}(N(A_{k_m})) \tag{4.10}$$

for all possible $k_1, \ldots, k_m \in \{0, 1, \ldots, n-1\}$.

Now, if $S \in M_{2^i}(\mathbb{C})$, $T \in M_{2^j}(\mathbb{C})$ with $j \leq i \leq n-1$ then

$$N(S) \operatorname{amp}_{j+1}^{i+1}(N(T)) = \begin{cases} 0 & \text{if } i = j \\ N\left(S \operatorname{amp}_{j+1}^{i}(N(T))\right) & \text{if } i > j. \end{cases}$$

So for S, $T \in M_{2^i}(\mathbb{C})$ and $U_p \in M_{2^{j(p)}}(\mathbb{C})$ with $j(p) < i \leq n$ $(p = 1, \ldots, k)$

$$\begin{aligned} &\operatorname{amp}_{i+1}^{n}(N(S)) \operatorname{amp}_{j(1)+1}^{n}(N(U_1)) \ldots \operatorname{amp}_{j(k)+1}^{n}(N(U_k)) \operatorname{amp}_{i+1}^{n}(N(T)) \\ &= \operatorname{amp}_{i+1}^{n}\left(N(S) \operatorname{amp}_{j(1)+1}^{n}(N(U_1)) \ldots \operatorname{amp}_{j(k)+1}^{n}(N(U_k)) N(T)\right) \\ &= \operatorname{amp}_{i+1}^{n}\left(N\left(S \operatorname{amp}_{j(1)+1}^{n}(N(U_1)) \ldots \operatorname{amp}_{j(k)+1}^{n}(N(U_k))\right) N(T)\right) \\ &= \operatorname{amp}_{i+1}^{n}(0) = 0 \end{aligned}$$

since $N(\cdot)N(\cdot) = 0$ for any feasible arguments. In particular, if

$$\operatorname{amp}_{k_i+1}^{n}(N(A_{k_i})) \operatorname{amp}_{k_{i+1}+1}^{n}\left(N(A_{k_{i+1}})\right) \ldots \operatorname{amp}_{k_j+1}^{n}\left(N(A_{k_j})\right)$$

is non-zero and has $k_i = k_j$ then $k_p > k_i$ for some p with $i < p < j$. It follows that a non-zero product of the form (4.10) can have at most one factor

$$D_k := \operatorname{amp}_{k+1}^{n}(N(A_k))$$

with $k = n-1$, two with $k = n-2$,… and 2^{n-1} with $k = 0$. However, since $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$, we see that such a product must have exactly these factors. Indeed these factors can only be written in one way so as to obtain a non-zero product. The two factors D_{n-2} must enclose D_{n-1} so the product is of the form

$$\ldots D_{n-2} \ldots D_{n-1} \ldots D_{n-2} \ldots,$$

the four factors D_{n-3} must pairwise enclose the D_{n-2} and D_{n-1} factors so the product is of the form

$$\ldots D_{n-3} \ldots D_{n-2} \ldots D_{n-3} \ldots D_{n-1} \ldots D_{n-3} \ldots D_{n-2} \ldots D_{n-3} \ldots,$$

and so on.

Thus there is only one non-zero product of the form (4.10), and that product is just the matrix X_n. □

The ordering of the factors of the one non-zero product mentioned above is easiest understood when we plot the indices of the factors, as in Figure 4.2. The concluding argument of the above proof may the be interpreted heuristically as 'points of equal height must enclose a higher point'.

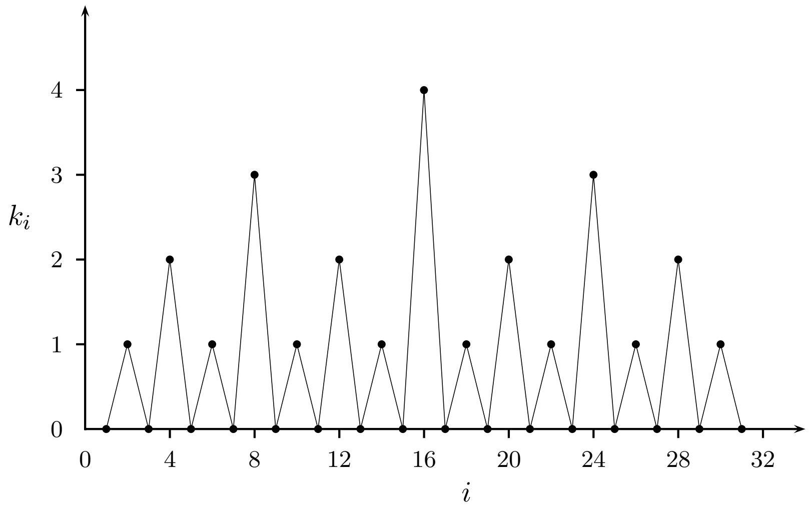

Figure 4.2.: Indices of the the factors of the only non-zero product (4.10) when $n = 3$. The lines at non-integer values are for clarity only.

Lemma 4.6.3. *For $n \in \mathbb{N}$*

$$X_n = \frac{(-1)^{[(n-1)/2]}}{2^n} N^n(1) \tag{4.11}$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}$, and consequently

$$\left\|B_n^{2^n-1}\right\|_{\mathrm{op}} = \|X_n\|_{\mathrm{op}} = \frac{1}{2^n}\|N^n(1)\|_{\mathrm{op}} = 1 \quad (n \in \mathbb{N}).$$

Proof. First note that for any matrices T_1, T_2 and T_3, of equal dimension

$$\begin{aligned} N(T_1)P(T_2)N(T_3) &= \begin{bmatrix} T_1 & -T_1 \\ T_1 & -T_1 \end{bmatrix} \begin{bmatrix} T_2 & T_2 \\ -T_2 & -T_2 \end{bmatrix} \begin{bmatrix} T_3 & -T_3 \\ T_3 & -T_3 \end{bmatrix} \\ &= 4N\left(T_1 T_2 T_3\right). \end{aligned} \tag{4.12}$$

Note also that

$$
\begin{aligned}
WP(WT) &= \begin{bmatrix} 0 & W \\ W & 0 \end{bmatrix} \begin{bmatrix} WT & WT \\ -WT & -WT \end{bmatrix} \\
&= \begin{bmatrix} -W^2T & -W^2T \\ W^2T & W^2T \end{bmatrix} \\
&= -P(T). \qquad (4.13)
\end{aligned}
$$

Using (4.12) and (4.13) we obtain, from the definition of the A_i,

$$
\begin{aligned}
N^n(1)A_nN^n(1) &= \frac{1}{2}N\left(N^{n-1}(1)\right)P\left(WA_{n-1}\right)N\left(N^{n-1}(1)\right) \\
&= \frac{1}{2}4N\left(N^{n-1}(1)WA_{n-1}N^{n-1}(1)\right) \\
&= 2\frac{1}{2}N\left(N^{n-1}(1)WP(WA_{n-2})N^{n-1}(1)\right) \\
&= -N\Big(N\left(N^{n-2}(1)\right)P(A_{n-2})N\left(N^{n-2}(1)\right)\Big) \\
&= -4N^2\left(N^{n-2}(1)A_{n-2}N^{n-2}(1)\right). \qquad (4.14)
\end{aligned}
$$

We now claim that

$$
N^n(1)A_nN^n(1) = (-1)^{[n/2]}2^{n-1}N^n(1) \quad (n \in \mathbb{N}), \qquad (4.15)
$$

which we prove by induction. That (4.15) holds for $n = 0$ and $n = 1$ is trivial, so suppose that for some $k \geq 2$, (4.15) holds for $n = 0, 1, \ldots, k$. We have

$$
\begin{aligned}
N^{k+1}(1)A_{k+1}N^{k+1}(1) &= -4N^2\left(N^{k-1}(1)A_{k-1}N^{k-1}(1)\right) \\
&= -4N^2\left((-1)^{[(k-1)/2]}2^{k-2}N^{k-1}(1)\right) \\
&= (-1)^{[(k+1)/2]}2^kN^{k+1}(1)
\end{aligned}
$$

which completes the inductive step, and so proves (4.15).

We can now finish the proof: suppose that

$$
X_k = \frac{(-1)^{[(k-1)/2]}}{2^k}N^k(1)
$$

holds for $k = 1, \ldots, n$. Then

$$
\begin{aligned}
X_{n+1} &= N(X_nA_nX_n) \\
&= \left(\frac{1}{2^n}\right)^2 N\left(N^n(1)A_nN^n(1)\right) \\
&= \frac{1}{2^{2n}}N\left((-1)^{[n/2]}2^{n-1}N^n(1)\right) \\
&= \frac{(-1)^{[n/2]}}{2^{n+1}}N^{n+1}(1).
\end{aligned}
$$

The lemma now follows by induction, since clearly (4.11) holds for $n = 1$. $\square$

To complete the proof of the theorem it only remains to show that $\|B_n\|_{\text{op}} \leq 1$ for $n \in \mathbb{N}$. Our argument uses the fact that $\|T\|_{\text{op}}^2 = \|T^*T\|_{\text{op}}$ for any matrix T, and requires the equalities

$$N(T)^* = P(T^*), \quad P(T)^* = N(T^*), \text{ and}$$
$$\operatorname{amp}_m^n(T)^* = \operatorname{amp}_m^n(T^*)$$

which follow directly from the definitions.

Lemma 4.6.4. *The inequality $\|B_n\|_{\text{op}} \leq 1$ obtains for $n \in \mathbb{N}$.*

Proof. We first show that

$$\|A_n^* A_n\|_{\text{op}} = 1/4 \tag{4.16}$$

holds for $n = 0, 1, \ldots$. The case $n = 0$ is just the definition, while if (4.16) holds for n then, since

$$\begin{aligned} A_{n+1}^* A_{n+1} &= \frac{1}{4}\left(P(WA_n)^* P(WA_n)\right) \\ &= \frac{1}{4} N(A_n^* W) P(WA_n) \\ &= \frac{1}{2}\begin{bmatrix} A_n^* A_n & A_n^* A_n \\ A_n^* A_n & A_n^* A_n \end{bmatrix}, \end{aligned}$$

we have

$$\left\|A_{n+1}^* A_{n+1}\right\|_{\text{op}} = \frac{1}{2}\left\|\begin{bmatrix} A_n^* A_n & A_n^* A_n \\ A_n^* A_n & A_n^* A_n \end{bmatrix}\right\|_{\text{op}} = \|A_n^* A_n\|_{\text{op}} = \frac{1}{4}.$$

Next note that for all feasible matrices S, T

$$\operatorname{amp}_{k-1}^k(S) P(T) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} T & T \\ -T & -T \end{bmatrix} = P(ST)$$

and

$$\operatorname{amp}_{k-1}^k(S) W P(T) = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} -WT & -WT \\ WT & WT \end{bmatrix} = -P(SWT).$$

Hence, for $k < n$

$$\begin{aligned}
&\operatorname{amp}^n_{k+1}(P(A_k^*))A_n \\
&= \operatorname{amp}^n_{n-1}\left(\operatorname{amp}^{n-1}_{k+1}(P(A_k^*))\right)\frac{1}{2}P(WA_{n-1}) \\
&= \frac{1}{2}P\left(\operatorname{amp}^{n-1}_{k+1}(P(A_k^*))WA_{n-1}\right) \\
&= \frac{1}{2}P\left(\operatorname{amp}^{n-1}_{n-2}\left(\operatorname{amp}^{n-2}_{k+1}(P(A_k^*))\right)W\frac{1}{2}P(WA_{n-2})\right) \\
&= \frac{1}{4}P\left(-P\left(\operatorname{amp}^{n-2}_{k+1}(P(A_k^*))W^2A_{n-2}\right)\right) \\
&= \frac{1}{4}P^2\left(\operatorname{amp}^{n-2}_{k+1}(P(A_k^*))A_{n-2}\right) \\
&\vdots \\
&= \begin{cases} 2^{k-n+1}P^{n-k-1}\left(\operatorname{amp}^{k+1}_{k+1}(P(A_k^*)A_{k+1})\right) & n-k+1 \text{ even} \\ 2^{k-n+1}P^{n-k-1}\left(\operatorname{amp}^{k+1}_{k+1}(P(A_k^*)WA_{k+1})\right) & n+k-1 \text{ odd} \end{cases} \\
&= \begin{cases} 2^{k-n+1}P^{n-k-1}\left(P(A_k^*)\frac{1}{2}P(WA_k)\right) & n-k+1 \text{ even} \\ 2^{k-n+1}P^{n-k-1}\left(P(A_k^*)\frac{1}{2}WP(WA_k)\right) & n-k+1 \text{ odd} \end{cases} \\
&= 0.
\end{aligned}$$

It follows that $B_k^*A_k = 0$ for each $k \in \mathbb{N}$, and a similar calculation shows that $B_kA_k^*$ is zero for each k. From these we also obtain

$$A_k^*B_k = (B_k^*A_k)^* = 0, \quad A_kB_k^* = (B_kA_k^*)^* = 0, \quad (k \in \mathbb{N}),$$

which enables us to finish the proof. We will show that either $\|B_n\|_{\mathrm{op}} \leq \|B_{n-1}\|_{\mathrm{op}}$ or $\|B_n\|_{\mathrm{op}} = 1$ and, since $\|B_1\|_{\mathrm{op}} = 1$, conclude that $\|B_n\|_{\mathrm{op}} \leq 1$ for all $n \in \mathbb{N}$.

Now,

$$\begin{aligned}
B_n^*B_n &= \left(\operatorname{amp}^n_{n-1}(B_{n-1}) + N(A_{n-1})\right)^*\left(\operatorname{amp}^n_{n-1}(B_{n-1}) + N(A_{n-1})\right) \\
&= \left(\operatorname{amp}^n_{n-1}(B_{n-1}^*) + P(A_{n-1}^*)\right)\left(\operatorname{amp}^n_{n-1}(B_{n-1}) + N(A_{n-1})\right) \\
&= \operatorname{amp}^n_{n-1}\left(B_{n-1}^*B_{n-1}\right) + \operatorname{amp}^n_{n-1}\left(B_{n-1}^*\right)N(A_{n-1}) \\
&\quad + P(A_{n-1}^*)\operatorname{amp}^n_{n-1}(B_{n-1}) + P(A_{n-1}^*)N(A_{n-1}) \\
&= \operatorname{amp}^n_{n-1}\left(B_{n-1}^*B_{n-1}\right) + N(B_{n-1}^*A_{n-1}) + P(A_{n-1}^*B_{n-1}) \\
&\quad + P(A_{n-1}^*)N(A_{n-1}) \\
&= \begin{bmatrix} B_{n-1}^*B_{n-1} & 0 \\ 0 & B_{n-1}^*B_{n-1} \end{bmatrix} + \\
&\quad + 0 + 0 + 2\begin{bmatrix} A_{n-1}^*A_{n-1} & -A_{n-1}^*A_{n-1} \\ -A_{n-1}^*A_{n-1} & A_{n-1}^*A_{n-1} \end{bmatrix}.
\end{aligned} \tag{4.17}$$

So, taking $v = [v_1,\, v_2]^T \in \mathbb{C}^{2^n}$ to be an eigenvector of $B_n^* B_n$ with

$$B_n^* B_n v = \|B_n^* B_n\|_{\text{op}}\, v \quad \left(= \|B_n\|_{\text{op}}^2\, v\right),$$

we obtain from (4.17) the pair

$$\begin{aligned} B_{n-1}^* B_{n-1} v_1 + 2A_{n-1}^* A_{n-1} v_1 - 2A_{n-1}^* A_{n-1} v_2 &= \|B_n\|_{\text{op}}^2\, v_1 \\ B_{n-1}^* B_{n-1} v_2 - 2A_{n-1}^* A_{n-1} v_1 + 2A_{n-1}^* A_{n-1} v_2 &= \|B_n\|_{\text{op}}^2\, v_2. \end{aligned}$$

Adding these and taking norms we find that

$$\|B_n\|_{\text{op}}^2 \|v_1 + v_2\| \le \left\|B_{n-1}^* B_{n-1}\right\|_{\text{op}} \|v_1 + v_2\| = \|B_{n-1}\|_{\text{op}}^2 \|v_1 + v_2\|$$

so that $\|B_n\|_{\text{op}} \le \|B_{n-1}\|_{\text{op}}$ provided $v_1 \ne -v_2$. In the case that $v_1 = -v_2$ we have

$$B_{n-1}^* B_{n-1} v_1 + 4A_{n-1}^* A_{n-1} v_1 = \|B_n\|_{\text{op}}^2\, v_1 \tag{4.18}$$

and so, multiplying these vectors by $B_{n-1}^* B_{n-1}$, we obtain

$$B_{n-1}^* B_{n-1} \left(B_{n-1}^* B_{n-1} v_1\right) = \|B_n\|_{\text{op}}^2\, B_{n-1}^* B_{n-1} v_1 - 4B_{n-1}^* B_{n-1} A_{n-1}^* A_{n-1} v_1.$$

Since the final term is zero we can take norms to see that

$$\|B_n\|_{\text{op}}^2 \left\|B_{n-1}^* B_{n-1} v_1\right\| \le \left\|B_{n-1}^* B_{n-1}\right\|_{\text{op}} \left\|B_{n-1}^* B_{n-1} v_1\right\|$$

so $\|B_n\|_{\text{op}} \le \|B_{n-1}\|_{\text{op}}$ in this case, unless $B_{n-1}^* B_{n-1} v_1 = 0$. If this final exception occurs we have

$$\|B_n\|_{\text{op}}^2\, v_1 = 4A_{n-1}^* A_{n-1} v_1$$

by (4.18), and so

$$\|B_n\|_{\text{op}}^2 = 4\left\|A_{n-1}^* A_{n-1}\right\|_{\text{op}} = 1$$

by (4.16) and noting that $v_1 \ne 0$ since $v = [v_1, -v_1]^T$ is assumed to be an eigenvector. $\square$

5. Related Properties

5.1. Introductory Remarks

We have seen in Section 3.4, that for some classes of Banach algebras, spectral uniformity and topologically bounded index coincide with other well-known finiteness properties: subhomogeneity, the satisfaction of a polynomial identity and injectivity in the sense of Varopoulos. In this chapter we aim to clarify these relationships a little, and in so doing describe some techniques and results that may be of interest in themselves. We begin with a summary of what is known.

For C^*-algebras subhomogeneity is equivalent to the satisfaction of a polynomial identity (Johnson [33, Prop. 6.1]) and recently it was shown that subhomogeneity is equivalent to injectivity (Aristov [3]) for C^*-algebras. Johnson (*ibid.*) also shows that a C^*-algebra A, satisfying a polynomial identity, has weak operator closure $\overline{A}^{\mathrm{w}}$ which is the finite sum of type I_k von Neumann algebras. By Proposition 3.4.1, $\overline{A}^{\mathrm{w}}$ is spectrally uniform and then, of course, A is too. These implications are illustrated in Figure 5.1.

For semisimple Banach algebras we retain the equivalence of subhomogeneity and the satisfaction of a polynomial identity, as can be seen by inspection of the the proof of the aforementioned result of Johnson. A semisimple Banach algebra satisfying a polynomial identity is always of topologically bounded index (Corollary 4.2.3) but may fail to be spectrally uniform (Example 3.1.4). We have previously mentioned that $\ell^1(FS_2)$ is of topologically bounded index (in Example 2.4.4) since it has no non-zero topologically nilpotent elements, and it is easy to see that it does not satisfy a polynomial identity. There are semisimple commutative Banach algebras which are not injective (for example $\ell^1(\mathbb{Z})$ as is shown in [8, Cor. 50.6]). Of course all such algebras are of topologically bounded index. We summarise these implications in Figure 5.2.

The questions remaining for such, and more general, algebras are the following.

1. Does the spectral uniformity of a Banach algebra imply that it satisfies a polynomial identity?

2. Does the injectivity of a Banach algebra imply that it satisfies the other finiteness properties of Figure 5.2?

The first question is an interesting possibility: were the implication to hold we would find an entirely algebraic property 'sandwiched' by spectral uniformity and

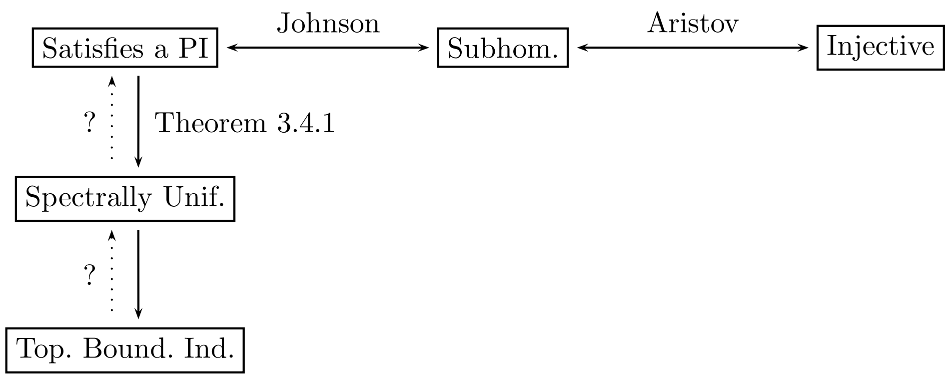

Figure 5.1.: Known implications for some finiteness properties in C^*-algebras

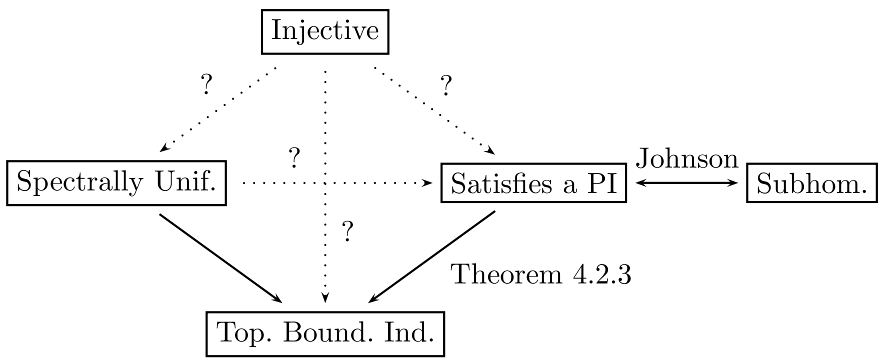

Figure 5.2.: Known implications for some finiteness properties in semisimple Banach algebras.

topologically bounded index. However, we have not been able to demonstrate such an implication, nor find a suitable counterexample.

We are able to make some progress on the second question by constructing a (non-semisimple) Banach algebra which is injective, but does not satisfy a polynomial identity. In so doing we develop some criteria for the injectivity of, amongst others, semigroup algebras. To describe these results we recall some of the theory of tensor products and give some detailed definitions which were promised earlier.

If A and B are Banach spaces with dual spaces A^* and B^* then for $a \in A$ and $b \in B$ we define $a \otimes b$ to be the bilinear form

$$\begin{aligned} a \otimes b : A^* \times B^* &\longrightarrow \mathbb{C} \\ (f,g) &\longmapsto f(a)g(b). \end{aligned}$$

With the natural co-ordinate-wise addition and scalar multiplication, the linear span of such forms is a linear space, which we denote $A \otimes B$. A norm on $A \otimes B$ which satisfies

$$\|a \otimes b\| = \|a\| \, \|b\| \quad (a \in A, \, b \in B)$$

is said to be a *cross norm*, one example of which is the *injective tensor norm* (also called the *weak tensor norm*) given by

$$\left\| \sum_{i,j=1}^{n} \lambda_{i,j} a_i \otimes b_j \right\|_\epsilon := \sup \left\{ \left| \sum_{i,j=1}^{n} \lambda_{i,j} f(a_i) g(b_j) \right| : \, f \in A_1^*, \, g \in B_1^* \right\}.$$

We will write $A \otimes_\epsilon B$ for $A \otimes B$ equipped with the injective tensor norm and $A \check{\otimes} B$ for the completion of $A \otimes_\epsilon B$. The notations $A \hat{\hat{\otimes}} B$, $A \overline{\otimes}_\epsilon B$ and $A \epsilon B$ are used in the literature to denote $A \check{\otimes} B$.

Another cross-norm is the *projective tensor norm*

$$\|u\|_\pi := \inf \left\{ \sum_{k,l=1}^{p} \|x_k\| \, \|y_l\| : \sum_{k,l=1}^{p} x_k \otimes y_l = u \right\}$$

and $A \otimes B$ equipped with $\|\cdot\|_\pi$ is denoted $A \otimes_\pi B$ while the completion is written as $A \hat{\otimes} B$. For a more detailed treatment of this approach to tensor products of Banach spaces we refer the reader to [8, §42].

For a Banach algebra A we will write R_A for the linearisation of the mapping

$$\begin{aligned} R_A : A \otimes_\epsilon A &\longrightarrow A \\ a \otimes b &\longmapsto ab \end{aligned}$$

and say that A is *injective* if R_A is bounded. Injective Banach algebras were introduced by Varopoulos in [50] and investigated by several authors in the 1970s. More recently the theory of completely bounded operators has renewed interest in the topic. Our deliberations do not deal with these matters but instead address some criteria for injectivity.

5.2. Necessary Conditions for Injectivity

Our necessary conditions for injectivity follow from the fact that the Banach algebra $\ell^1(\mathbb{N})$, with convolution product $*$, is not injective [8, Cor. 50.6]. In essence we argue that a Banach algebra with an increasing sequence of certain subspaces which are isometric to finite dimensional subspaces of $\ell^1(\mathbb{N})$, cannot be injective.

Proposition 5.2.1. *Let A be a Banach algebra and suppose that there is an increasing sequence (n_i) of natural numbers, a sequence (a_i) of elements of A and that*

$$\left\| \sum_{k=1}^{n_i} \lambda_{i,k} a_i^k \right\| = \sum_{k=1}^{n_i} |\lambda_{i,k}| \quad (\lambda_{i,k} \in \mathbb{C}). \tag{5.1}$$

Then A is not injective.

Proof. We take $M > 0$ to be fixed. Since $\ell^1(\mathbb{N})$ is not injective there is some $u \in \ell^1(\mathbb{N}) \otimes \ell^1(\mathbb{N})$ with

$$\left\| R_{\ell^1(\mathbb{N})}(u) \right\|_1 > M \left\| u \right\|_\epsilon .$$

Indeed we may, and do, suppose that u has a finite representation

$$u = \sum_{j=1}^{n} x_j \otimes y_j$$

where, relative to the natural basis (e_i) of $\ell^1(\mathbb{N})$,

$$x_j = \sum_{k=1}^{N} \alpha_{j,k} e_k \quad \text{and} \quad y_j = \sum_{k=1}^{N} \beta_{j,k} e_k$$

since such elements are easily seen to be dense in $\ell^1(\mathbb{N}) \check{\otimes} \ell^1(\mathbb{N})$.

Now, using the hypothesis of the proposition, take i to be large enough to ensure that $n_i \geq 2N$. Clearly (5.1) implies that a_i, $a_i^2, \ldots, a_i^{2N}$ are linearly independent. Define a linear mapping by

$$\begin{aligned} \Psi : \ell^1(\mathbb{N}) &\longrightarrow A \\ e_k &\longmapsto a_i^k \end{aligned}$$

noticing that the restriction of Ψ to span $\{e_i, \dots, e_{2N}\}$ is an isometry. Now

$$\begin{aligned}
&\left\| \sum_{j=1}^{n} \Psi(x_j) \otimes \Psi(x_j) \right\|_\epsilon \\
&= \left\| \sum_{j=1}^{n} \sum_{k=1}^{N} \sum_{l=1}^{N} \alpha_{j,k}\beta_{j,l} \Psi(e_k) \otimes \Psi(e_l) \right\|_\epsilon \\
&= \left\| \sum_{k=1}^{N} \sum_{l=1}^{N} \left(\sum_{j=1}^{n} \alpha_{j,k}\beta_{j,l} \right) \Psi(e_k) \otimes \Psi(e_l) \right\|_\epsilon \\
&= \sup \left\{ \left| \sum_{k=1}^{N} \sum_{l=1}^{N} \left(\sum_{j=1}^{n} \alpha_{j,k}\beta_{j,l} \right) f\left(\Psi(e_k)\right) g\left(\Psi(e_l)\right) \right| : f, g \in A_1^* \right\} \\
&\leq \sup \left\{ \left| \sum_{k=1}^{N} \sum_{l=1}^{N} \left(\sum_{j=1}^{n} \alpha_{j,k}\beta_{j,l} \right) F\left(e_k\right) G\left(e_l\right) \right| : F, G \in \left(\ell^1(\mathbb{N})\right)_1^* \right\} \\
&= \left\| \sum_{j=1}^{n} x_j \otimes y_j \right\|_\epsilon \qquad (5.2)
\end{aligned}$$

since the restrictions of $f \circ \Psi$ and $g \circ \Psi$ to span $\{e_1, \dots, e_{2N}\}$ are linear functionals with norm no greater than one, and so may be extended to functionals on $\ell^1(\mathbb{N})$ with norm no greater than one by Hahn-Banach.

Now note that

$$\begin{aligned}
\Psi(x_j)\Psi(y_j) &= \left(\sum_{k=1}^{n} \alpha_{j,k} a_i^k \right) \left(\sum_{k=1}^{n} \beta_{j,k} a_i^k \right) \\
&= \Psi(x_j * y_j) \qquad (j = 1, \dots, n)
\end{aligned}$$

so writing

$$v = \sum_{j=1}^{n} \Psi(x_j) \otimes \Psi(y_j) \in A \check{\otimes} A$$

we have

$$\begin{aligned}
R_A(v) &= \sum_{j=1}^{n} \Psi(x_j)\Psi(y_j) \\
&= \sum_{j=1}^{n} \Psi(x_j * y_j) \\
&= \Psi\left(R_{\ell^1(\mathbb{N})}(u)\right)
\end{aligned}$$

and so

$$\begin{aligned} \|R_A(v)\| &= \left\|\Psi\left(R_{\ell^1(\mathbb{N})}(u)\right)\right\| \\ &= \left\|R_{\ell^1(\mathbb{N})}(u)\right\| \\ &\geq M\,\|u\|_\epsilon \\ &\geq M\,\|v\|_\epsilon\,, \end{aligned}$$

the final inequality being just (5.2). Since M is arbitrary we conclude that R_A is not bounded, as required. □

In particular, Proposition 5.2.1 leads quickly to a restriction that semigroups S must satisfy if the semigroup algebra $\ell^1(S)$ is to be injective.

Corollary 5.2.2. *Suppose that S is a semigroup and that $\ell^1(S)$ is injective. Then there is a number N such that*

$$\operatorname{card}\left\{s, s^2, \ldots\right\} \leq N \quad (s \in S)$$

and so, in particular, such a semigroup is periodic.

Notice that if $\ell^1(S)$ is injective then S satisfies the conditions of Proposition 4.4.1 which are required if $\ell^1(S)$ is to be of topologically bounded index. Thus these criteria alone will not help us find an injective $\ell^1(S)$ which is not of topologically bounded index (should such an algebra exist).

5.3. Sufficient Conditions for Some Semigroup Algebras to be Injective

In this section we describe a condition on a countable semigroup S which forces some algebras constructed on it to be injective. This condition involves 'most' products of semigroup elements being 'zero'. We must tread carefully with our notation here since, as we have previously mentioned, the zero θ of a semigroup S is not zero as an element of $\mathbb{C}[S]$. We make the following (notationally non-standard) definition of a natural construction of a Banach algebra from a semigroup with zero. This definition may be thought of as placing the well-known 'generators and relations Banach algebra' construction in the notational context of semigroup algebras.

Definition 5.3.1. Suppose that S is a semigroup with zero θ and that ω is a function $S\backslash\{\theta\} \to (0, \infty)$ satisfying

$$\omega(st) \leq \omega(s)\omega(t) \quad (s,\, t \in S\backslash\{\theta\}).$$

$$\begin{array}{c|cccc}
\cdot & \theta & e_1 & e_2 & \cdots \\
\hline
\theta & \theta & \theta & \theta & \cdots \\
e_1 & \theta & e_1 & \theta & \cdots \\
e_2 & \theta & \theta & e_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Figure 5.3.: Cayley diagram of a semigroup S related to ℓ^1

Then we will say that ω is an *algebra θ-weight* on $S\backslash\{\theta\}$. We will write $\mathbb{C}_\theta[S]$ for the space of finite formal sums in non-zero semigroup elements, which is an algebra with the product

$$(1_{\mathbb{C}}s)(1_{\mathbb{C}}t) = \begin{cases} 1_{\mathbb{C}}st & st \neq \theta \\ 0_{\mathbb{C}} & st = \theta \end{cases}$$

so that a typical element is of the form

$$x = \sum_{s \in S\backslash\{\theta\}} \lambda_s s$$

where only finitely many of the $\lambda_s \in \mathbb{C}$ are non-zero. Notice that $\mathbb{C}_\theta[S]$ is isomorphic to the algebraic quotient $\mathbb{C}\,[S]/\mathbb{C}\,[\theta]$. The *$\theta$-weighted semigroup algebra* of S, denoted $\ell^1_\theta(S,\omega)$, is the completion of $\mathbb{C}_\theta[S]$ in the *θ-weighted ℓ^1 norm*

$$\|x\|_{1,\omega} := \sum_{s \in S\backslash\{\theta\}} |\lambda_s|\omega(s).$$

If 1 denotes the unit weight on a semigroup S, then we write $\ell^1_\theta(S)$ for $\ell^1_\theta(S,1)$.

If S is a semigroup with zero and ω is an algebra weight on S, then clearly the restriction of ω to $S\backslash\{\theta\}$ is a algebra θ-weight and

$$\ell^1_\theta(S,\omega) \overset{1}{\cong} \ell^1(S,\omega)/\mathbb{C}\,[\theta].$$

As an example: if S is the semigroup $\{\theta, e_1, e_2, \ldots\}$ with a Cayley diagram as in Figure 5.3, and 1 denotes the unit weight then

$$\ell^1_\theta(S,1) \overset{1}{\cong} \ell^1(S,1)/\mathbb{C}\,[\theta] \overset{1}{\cong} \ell^1.$$

However not every θ-weight arises as a restriction of a weight on a semigroup with zero, since such a weight necessarily has $\omega(s) \geq 1$ for each $s \in S$. What will be important in the following is the fact that $\ell^1_\theta(S,\omega)$ is isometric, *as a Banach space*, to the space $\ell^1(S\backslash\{\theta\},\omega)$. Thus the dual of $\ell^1_\theta(S,\omega)$ is isometric (as a Banach space) with $\ell^\infty(S\backslash\{\theta\},\omega^{-1})$; the completion of the space $\mathbb{C}\,[S\backslash\{\theta\}]$ in the norm

$$\left\| \sum_{s\in S\backslash\{\theta\}} \lambda_s s \right\|_{\infty,\omega^{-1}} = \sup_{s\in S\backslash\{\theta\}} |\lambda_s|\omega(s)^{-1}.$$

We need a short description of the duality theory of tensor products (we refer the reader to [11] for a detailed treatment) to introduce our notation. Suppose that A and B are Banach spaces. If $F \in A^* \otimes B^*$ has a representation

$$F = \sum_{i=1}^{n} f_i \otimes g_i \tag{5.3}$$

then we define

$$\begin{array}{rcl} \widetilde{F} : A \otimes_\epsilon B & \longrightarrow & \mathbb{C} \\ \displaystyle\sum_{j=1}^{m} a_j \otimes b_j & \longmapsto & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} f_i(a_j)g_i(b_j). \end{array}$$

From the definition of the tensor product one quickly sees that $\widetilde{F}$ is independent of the choice of the representation (5.3). It is also easy to see that the mapping $F \mapsto \widetilde{F}$ is an injection, and if

$$u = \sum_{j=1}^{m} a_j \otimes b_j \in A \otimes_\epsilon B$$

then

$$\begin{aligned} \left|\widetilde{F}(u)\right| &= \left|\sum_{i=1}^{n}\sum_{j=1}^{m} f_i(a_j)g_i(b_j)\right| \\ &\leq \sum_{i=1}^{n}\left|\sum_{j=1}^{m} f_i(a_j)g_i(b_j)\right| \\ &\leq \sum_{i=1}^{n} \|f_i\|\,\|g_i\|\,\|u\|_\epsilon \end{aligned}$$

and since this holds for any representation (5.3) we can take infima to obtain

$$\left|\widetilde{F}(u)\right| \leq \|F\|_\pi\,\|u\|_\epsilon \quad (u \in A \otimes_\epsilon B). \tag{5.4}$$

Thus the mapping $F \mapsto \widetilde{F}$ is a norm reducing injection $A^* \otimes_\pi B^* \hookrightarrow (A \otimes_\epsilon B)^*$.

We can now prove a lemma on which the results of this section rely. The proof is, in essence, a weighted and generalised version of Varopoulos's argument showing that ℓ^1 is injective (this is proved in [50] and attributed there to S. Kaijser).

Lemma 5.3.2. *Let S be a countable semigroup with zero θ, say*

$$S = \{\theta, e_1, e_2, \ldots\},$$

and suppose that ω is an algebra θ-weight on $S \backslash \{\theta\}$. If

$$u = \sum_{i,j=1}^{m} \lambda_{i,j} e_i \otimes e_j \in \ell^1_\theta(S, \omega) \otimes_\epsilon \ell^1_\theta(S, \omega)$$

and σ is a permutation on $\{1, \ldots, m\}$, then

$$\sum_{i=1}^{n} \left| \lambda_{i,\sigma(i)} \omega(i) \omega(\sigma(i)) \right| \leq \|u\|_\epsilon .$$

Proof. We define a function h by

$$h(i) = \begin{cases} 0 & \text{if } \lambda_{i,\sigma(i)} = 0 \\ \overline{\operatorname{sgn}(\lambda_{i,\sigma(i)})} & \text{if } \lambda_{i,\sigma(i)} \neq 0 \end{cases} \quad (i = 1, \ldots, m)$$

and identify $\{e_1, \ldots, e_m\}$ (as a set) with the group G of integers (mod m) so that h acts on $\{e_1, \ldots, e_m\}$ in a natural way. Write $\hat{G}$ for the group of characters on G (see [46, §12.2]) and define $F \in \ell^\infty(G, \omega^{-1}) \otimes_\pi \ell^\infty(G, \omega^{-1})$ by

$$F = \sum_{\chi \in \hat{G}} \left(\frac{1}{m} . h . \omega . \chi \right) \otimes \left(\omega . \overline{(\chi \circ \sigma^{-1})} \right)$$

where the point denotes pointwise multiplication and the circle, composition. Then $\widetilde{F}$ is a linear functional on $\ell^1(G, \omega) \otimes_\epsilon \ell^1(G, \omega)$ which we extend to a homonymous linear functional with the same norm on $\ell^1(S \backslash \{\theta\}, \omega) \otimes_\epsilon \ell^1(S \backslash \{\theta\}, \omega)$ by Hahn-Banach.

Thus with the aforementioned identification

$$\widetilde{F}(e_i \otimes e_j) = \frac{1}{m} h(i) \omega(i) \omega(j) \sum_{\chi \in \hat{G}} \chi(i) \overline{\chi\left(\sigma^{-1}(j)\right)}$$

and since

$$\sum_{\chi \in \hat{G}} \chi(g_1) \overline{\chi(g_2)} = \begin{cases} \operatorname{card}(G) = m & g_1 = g_2 \\ 0 & g_1 \neq g_2 \end{cases}$$

for $g_1,\ g_2 \in G$ we have

$$\widetilde{F}(u) = \sum_{i=1}^{m} \left|\lambda_{i,\sigma(i)}\right| \omega(i)\omega(\sigma(i)).$$

The required inequality now follows for

$$\begin{aligned} \left|\widetilde{F}(u)\right| &\leq \|F\|_{\pi}\, \|u\|_{\epsilon} \\ &\leq \sum_{\chi\in\hat{G}} \left\| \left(\frac{1}{m}.h.\omega.\chi\right) \right\|_{\infty,\omega^{-1}} \left\| \left(\omega.\overline{(\chi\circ\sigma^{-1})}\right) \right\|_{\infty,\omega^{-1}} \|u\|_{\epsilon} \\ &= \|u\|_{\epsilon}\,. \end{aligned}$$

□

Out first application of Lemma 5.3.2 is to construct an injective Banach algebra which does not satisfy a polynomial identity.

Example 5.3.3. Let $e_{i,j}$ denote the infinite matrix with one as the i,j-th entry and zeros elsewhere, and θ the infinite matrix of zeros. Then with the usual matrix multiplication the set

$$S := \{e_{i,j} : 1 \leq i < j\} \cup \{\theta\}$$

is a semigroup with zero. Define a weight ω on $S\backslash\{\theta\}$ by

$$\omega(i,j) = 2^{-(j-i)^2} \quad (1 \leq i < j).$$

To see that this is an algebra θ-weight note that

$$\begin{aligned} \omega(i,j)\omega(j,k) &= 2^{-(j-i)^2-(k-j)^2} \\ &= 2^{2(j-i)(k-j)}\omega(i,k) \end{aligned}$$

and so, by a short calculation,

$$\begin{aligned} \omega(i,k) &\leq 2^{-2(j-i)(k-j)}\omega(i,j)\omega(j,k) \\ &\leq 2^{-2(k-i-1)}\omega(i,j)\omega(j,k). \end{aligned}$$

Proposition 5.3.4. *With S and ω defined as above, the Banach algebra $\ell^1_\theta(S,\omega)$ is injective.*

Proof. Suppose that $u \in \ell^1_\theta(S,\omega) \otimes_\epsilon \ell^1_\theta(S,\omega)$ is of the form

$$u = \sum_{i<j,\, k<l} \lambda_{i,j,k,l} e_{i,j} \otimes e_{k,l}$$

where only finitely many of the $\lambda_{i,j,k,l}$ are non-zero. Then

$$\begin{aligned} R_{\ell^1_\theta(S,\omega)}(u) &= \sum_{i<j<l} \lambda_{i,j,j,l} e_{i,l} \\ &= \sum_{m=2}^{\infty} \sum_{i<j<i+m} \lambda_{i,j,j,i+m} e_{i,i+m} \end{aligned}$$

so that

$$\begin{aligned} \left\| R_{\ell^1_\theta(S,\omega)}(u) \right\|_{1,\omega} &= \sum_{m=2}^{\infty} \sum_{i<j<i+m} |\lambda_{i,j,j,i+m}| \omega(i, i+m) \\ &\leq \sum_{m=2}^{\infty} \sum_{i<j<i+m} |\lambda_{i,j,j,i+m}| 2^{-2(m-1)} \omega(i,j)\omega(j, i+m) \\ &= \sum_{m=2}^{\infty} 2^{-2(m-1)} \sum_{i<j<i+m} |\lambda_{i,j,j,i+m}| \omega(i,j)\omega(j, i+m). \quad (5.5) \end{aligned}$$

Now, with m fixed, for each pair (i,j) there is exactly one pair (k,l) such that $\lambda_{i,j,k,l}$ occurs in the inner sum of (5.5). Thus, by a suitable relabelling of the semigroup elements $e_{i,j}$, we can apply Lemma 5.3.2 to obtain

$$\sum_{i<j<i+m} |\lambda_{i,j,j,i+m}| \omega(i,j)\omega(j, i+m) \leq \|u\|_\epsilon \quad (m = 2, 3, \ldots)$$

and so from (5.5)

$$R_{\ell^1_\theta(S,\omega)}(u) \leq \sum_{m=2}^{\infty} 2^{-2(m-1)} \|u\|_\epsilon = \frac{1}{3} \|u\|_\epsilon .$$

The result now follows since elements of the form u (i.e. those with finite support) are dense in $\ell^1_\theta(S,\omega)\check{\otimes}\ell^1_\theta(S,\omega)$. □

The Cayley diagram of the semigroup S is shown in Figure 5.4.

It remains to show that $\ell^1_\theta(S,\omega)$ does not satisfy a polynomial identity. This can be achieved using half of 'Kaplansky's staircase'. If an algebra (not even necessarily normed) satisfies a polynomial identity then it satisfies a homogeneous multilinear identity of no greater degree [27, Lemma 6.2.4], so it suffices to show that $\ell^1_\theta(S,\omega)$ does not satisfy the identity

$$p(X_1, \ldots, X_n) := X_1 \ldots X_n + \sum_{\sigma \neq 1} \lambda_\sigma X_{\sigma(1)} \ldots X_{\sigma(n)}$$

$\cdot$	e_{12}	e_{13}	e_{14}	$\cdots$	e_{23}	e_{24}	e_{25}	$\cdots$	e_{34}	e_{35}	e_{35}	$\cdots$	e_{45}	e_{46}	e_{47}	$\cdots$	$\cdots$
e_{12}					e_{13}	e_{14}	e_{15}	$\cdots$									
e_{13}									e_{14}	e_{15}	e_{16}	$\cdots$					
e_{14}													e_{15}	e_{16}	e_{17}	$\cdots$	
$\vdots$																	$\ddots$
e_{23}									e_{24}	e_{25}	e_{26}	$\cdots$					
e_{24}													e_{25}	e_{26}	e_{27}	$\cdots$	
$\vdots$																	$\ddots$
e_{34}													e_{35}	e_{36}	e_{37}	$\cdots$	
$\vdots$																	$\ddots$
$\vdots$																	$\ddots$

Figure 5.4.: Cayley diagram of the semigroup S described in Example 5.3.3. Blanks denote the zero θ and the shading indicates in which of the inner summations in (5.5) the element occurs;
e_{ij} is in the summation with $m = 2$,
e_{ij} is in the summation with $m = 3$ and
e_{ij} is in the summation with $m = 4$.

where the summation is over all non-trivial permutations on $\{1, \ldots, n\}$. But this is obvious since

$$p(e_{1,2}, e_{2,3}, \ldots, e_{n,n-1}) = e_{1,n+1} \neq 0.$$

We have previously mentioned that a corollary to Aristov's result [3] is that a C^*-algebra is injective if, and only if, it satisfies a polynomial identity. Example 5.3.3 shows that this characterization does not extend to arbitrary Banach algebras. Unfortunately this example is not semisimple. Indeed, since the ideals

$$A_k := \operatorname{span}\{e_{i,j} : i \leq k,\ j \in \mathbb{N}\} \lhd \ell^1_\theta(S, \omega)$$

are nil and their union is dense, $\ell^1_\theta(S, \omega)$ is radical.

Our second application of Lemma 5.3.2 demonstrates that $\ell^1_\theta(S, \omega)$ is injective if most of the products of semigroup elements are zero.

Proposition 5.3.5. *Let S be a countable semigroup with zero θ and let ω be an algebra θ-weight on $S \backslash \{\theta\}$. Suppose further that there is some $K \in \mathbb{N}$ such that for each non-zero $s \in S$ there are at most K distinct $t \in S$ with $st \neq \theta$, and at most K distinct $r \in S$ with $rs \neq \theta$. Then $\ell^1_\theta(S, \omega)$ is injective and*

$$\left\| R_{\ell^1_\theta(S,\omega)} \right\| \leq K^2.$$

Proof. As before we set $S = \{\theta, e_1, e_2, \ldots\}$ and suppose $u \in \ell^1_\theta(S,\omega) \otimes \ell^1_\theta(S,\omega)$ is of the form

$$u = \sum_{i,j=1}^{m} \lambda_{i,j} e_i \otimes e_j$$

so that

$$R_{\ell^1_\theta(S,\omega)}(u) = \sum_{i,j=1}^{m} \lambda_{i,j} e_i e_j.$$

Let $M = \max\{n : \, e_i e_j = e_n, \, i,j = 1,\ldots,m\}$ so that we can write

$$R_{\ell^1_\theta(S,\omega)}(u) = \sum_{k=1}^{M} \left(\sum_{e_i e_j = e_n} \lambda_{i,j} \right) e_k$$

and then

$$\begin{aligned} \left\| R_{\ell^1_\theta(S,\omega)}(u) \right\| &= \sum_{k=1}^{M} \left| \sum_{e_i e_j = e_n} \lambda_{i,j} \right| \omega(k) \\ &\leq \sum_{\substack{1 \leq i,j \leq M \\ e_i e_j \neq \theta}} |\lambda_{i,j}| \omega(i) \omega(j). \end{aligned} \tag{5.6}$$

We now claim that one can write the sum in (5.6) as the sum of at most K^2 sums of the form

$$\sum_{p=1}^{q} \left| \lambda_{i_p, j_p} \right| \omega(i_p) \omega(j_p) \tag{5.7}$$

where $i_1 < i_2 < \cdots < i_q$ and the j_p are distinct. To see this we first show that we can write the sum in (5.6) as the sum of at most K sums of the form (5.7) with $i_1 < i_2 < \cdots < i_q$ (i.e. the j_p need *not* be distinct). To find these take $i_{1,1}$ with $1 \leq i_{1,1}$ to be minimal such that $e_{i_{1,1}} e_j \neq \theta$ for some j, and $j_{1,1}$ to be the minimal such j. Say that the pair $(i_{1,1}, j_{1,1})$ is *chosen.* Take $i_{1,2}$ to be minimal such that $i_{1,1} < i_{1,2}$ and $e_{i_{1,2}} e_j \neq \theta$ for some j, and $j_{1,2}$ to be the minimal such j. Say that the pair $(i_{1,2}, j_{1,2})$ is chosen. Proceeding in this fashion one obtains a sequence

$$(i_{1,1}, j_{1,1}), \, (i_{1,2}, j_{1,2}), \ldots, (i_{1,m(1)}, j_{1,m(1)})$$

While there is a non-chosen pair (i,j) with $e_j e_i \neq \theta$ we find the n-th sequence similarly. Take $i_{n,1}$ with to be minimal such that $e_{i_{n,1}} e_j \neq \theta$ for some j and such that $(i_{n,1}, j)$ is not chosen, and $j_{n,1}$ to be the minimal such j. Say that the pair $(i_{n,1}, j_{n,1})$ is chosen. Continuing in this way one obtains the n-th sequence

$$(i_{n,1}, j_{n,2}), \, (i_{n,2}, j_{n,2}), \ldots, (i_{n,m(n)}, j_{n,m(n)}) \tag{5.8}$$

of chosen pairs.

By hypothesis, this procedure terminates after we have have constructed at most K sequences of the form (5.8). Then for the n-th of these sequences one writes $i_p = i_{n,p}$, $j_p = j_{n,p}$ and $q = m(n)$ to obtain at most K sums of the form (5.7) with $i_1 < \cdots < i_q$. Clearly each of these sums can be written as the sum of at most K sums with the additional property that the j_p are distinct, and so one can write the sum in (5.6) as the sum of at most K^2 sums of the form claimed.

We can now complete the proof. By Lemma 5.3.2 each of the sums of the form (5.7) has

$$\sum_{p=1}^{q} \left|\lambda_{i_p,j_p}\right| \omega(i_p)\omega(j_p) \leq \|u\|_\epsilon$$

and since there are no more than K^2 such sums in the sum (5.6) we have

$$\left\|R_{\ell^1_\theta(S,\omega)}(u)\right\| \leq K^2 \|u\|_\epsilon .$$

Since this estimate holds for all u with finite support it extends to the closure and provides the bound on $\left\|R_{\ell^1_\theta(S,\omega)}\right\|$ which was claimed. □

In particular, Proposition 5.3.5 shows that ℓ^1 is injective with $\|R_{\ell^1}\| \leq 1$. Thus we recover the result (and the implicit bound) obtained by Varopoulos *ibid.*

We can extend the above proposition to show that the same condition on a semigroup S forces $\ell^1(S)$ to be injective.

Lemma 5.3.6. *Suppose that S is a countable semigroup with zero θ and that $\ell^1_\theta(S)$ is injective. Then $\ell^1(S)$ is injective with*

$$\left\|R_{\ell^1(S)}\right\| \leq 6\left\|R_{\ell^1_\theta(S)}\right\| + 1.$$

Proof. We will write $S = \{e_0, e_1, \ldots\}$, where $e_0 = \theta$, for simplicity of notation. If

$$u = \sum_{i,j=0}^{m} \lambda_{i,j} e_i \otimes e_j \in \ell^1(S) \otimes \ell^1(S)$$

then

$$\left|\sum_{i,j=0}^{m} \lambda_{i,j}\right| \leq \|u\|_\epsilon$$

and since

$$\sum_{e_ie_j=e_0} \lambda_{i,j} = \sum_{k=0}^{\infty}\left(\sum_{e_ie_j=e_k} \lambda_{i,j}\right) - \sum_{k=1}^{\infty}\left(\sum_{e_ie_j=e_k} \lambda_{i,j}\right)$$

we find that

$$\begin{aligned}\left|\sum_{e_ie_j=e_0}\lambda_{i,j}\right| &\leq \left|\sum_{i,j=0}^{m}\lambda_{i,j}\right| + \sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\lambda_{i,j}\right| \\ &\leq \|u\|_\epsilon + \sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\lambda_{i,j}\right|\end{aligned}$$

which gives

$$\begin{aligned}\left\|R_{\ell^1(S)}(u)\right\|_1 &= \sum_{k=0}^{\infty}\left|\sum_{e_ie_j=e_k}\lambda_{i,j}\right| \\ &\leq \|u\|_\epsilon + 2\sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\lambda_{i,j}\right|. \end{aligned} \tag{5.9}$$

Now

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\lambda_{i,j}\right| &= \left\|R_{\ell^1_\theta(S)}\left(\sum_{i,j=1}^{m}\lambda_{i,j}e_i\otimes e_j\right)\right\|_1 \\ &\leq \left\|R_{\ell^1_\theta(S)}\right\|\left\|\sum_{i,j=1}^{m}\lambda_{i,j}e_i\otimes e_j\right\|_{\ell^1_\theta(S)\otimes_\epsilon\ell^1_\theta(S)} \\ &= \left\|R_{\ell^1_\theta(S)}\right\|\left\|\sum_{i,j=1}^{m}\lambda_{i,j}e_i\otimes e_j\right\|_{\ell^1(S)\otimes_\epsilon\ell^1(S)}, \end{aligned} \tag{5.10}$$

since injective tensor products preserve subspaces [11, §4.3], and since

$$\sum_{i,j=1}^{m}\lambda_{i,j}e_i\otimes e_j = u - \left(\sum_{i=0}^{m}\lambda_{i,0}e_i\right)\otimes e_0 - e_0\otimes\left(\sum_{j=1}^{m}\lambda_{0,j}e_j\right)$$

we have

$$\begin{aligned}\left\|\sum_{i,j=1}^{m}\lambda_{i,j}e_i\otimes e_j\right\|_\epsilon &\leq \|u\|_\epsilon + \|e_0\|_1\left(\left\|\sum_{i=0}^{m}\lambda_{i,0}e_i\right\|_1 + \left\|\sum_{j=1}^{m}\lambda_{0,j}e_j\right\|_1\right) \\ &= \|u\|_\epsilon + \sum_{i=0}^{m}|\lambda_{i,0}| + \sum_{j=1}^{m}|\lambda_{0,j}| \\ &\leq 3\|u\|_\epsilon. \end{aligned} \tag{5.11}$$

Combining the inequalities (5.9), (5.10) and (5.11) now gives the bound

$$\left\|R_{\ell^1(S)}(u)\right\|_1 \leq 6\left\|R_{\ell^1_\theta(S)}(u)\right\|_1 + \|u\|_1$$

for elements u with finite support. This bound extends to the closure and so proves the proposition. $\square$

Corollary 5.3.7. *Let S be a countable semigroup with zero θ and suppose that there is some $K \in \mathbb{N}$ such that for each non-zero $s \in S$ there are at most K distinct $t \in S$ with $st \neq \theta$, and at most K distinct $r \in S$ with $rs \neq \theta$. Then $\ell^1(S)$ is injective with*

$$\left\|R_{\ell^1_\theta(S)}\right\| \leq 6K^2 + 1.$$

There are technical difficulties in proving a weighted version of Lemma 5.3.2 and for this reason we cannot present a weighted version of its corollary. However weights on a semigroup with zero are somewhat restricted, so such a weighted version would not be as interesting an extension as at first appears.

We conclude by mentioning that Corollaries 5.3.7 and 5.2.2 do not both characterize semigroups S (with zero) such that $\ell^1(S)$ is injective. Consider the semigroup S with elements $\theta, e_1, e_2, \ldots$ and product

$$e_i e_j = e_{\max\{i,j\}} \quad (i, j \geq 1).$$

Then every element is idempotent and so the necessary conditions of Corollary 5.2.2 are met. However

$$e_1 e_j = e_j \neq \theta \quad (j = 1, 2, \ldots)$$

so the sufficient conditions of Corollary 5.3.7 are not.

5.4. Sundry Results on Injectivity

In attempting to resolve some of the questions addressed in the previous sections one naturally investigates the consequences of injectivity. One well-known fact is that an injective commutative Banach algebra is necessarily a Q-algebra [50]. More generally, an injective Banach algebra is an *operator algebra:* isomorphic with a norm-closed subalgebra of $\mathscr{B}(\mathscr{H})$ (this a consequence of a result of Tonge [49, Th. 1′]). We show that this latter fact can be obtained, by elementary means, from the theorem [50, 2.1.ii)] and the criterion [51, §1], both of which are due to Varopoulos.

Recall that a *Radon measure* μ is a complex measure on a σ-algebra of subsets of a locally compact space, finite on compact subsets and inner-regular in the sense that

$$|\mu|(B) = \sup\{\, |\mu|(C) \colon C \subseteq B \text{ is compact} \,\}.$$

Theorem 5.4.1 (Varopoulos, 1972)**.** *Let A be an injective Banach algebra. Then there is a constant K such that for any $n \geq 1$ and $f \in A_1^*$, there is a Radon measure μ on $A_1^* \times \cdots A_1^*$ (n copies) with $\|\mu\| \leq K^{n-1}$ and*

$$f(a_1 \dots a_n) = \int_{A_1^* \times \cdots A_1^*} g_1(a_1) \dots g_n(a_n) d\mu(g_1, \dots, g_n)$$

for $a_1, \dots, a_n \in A$.

Theorem 5.4.2 (Varopoulos, 1975)**.** *A Banach algebra A is an operator algebra if there is $K > 0$ such that the following holds. For any $f \in A_1^*$, any $n \geq 1$ and any finite dimensional subspace $B \subseteq A$ we can find linear mappings*

$$L_i : B \longrightarrow \mathscr{B}(\mathscr{H}) \quad (i = 1, \dots, n)$$

(where $\mathscr{H}$ depends on B, n and f), and $F, G \in \mathscr{H}_1$ satisfying

$$\|L_i(a)\|_{\mathrm{op}} \leq K \|a\| \quad (a \in B,\ i = 1, \dots, n)$$

and

$$f(a_1, \dots, a_n) = \langle L_1(a_1) \dots L_n(a_n) F, G \rangle \quad (a_i \in B,\ i = 1, \dots, n).$$

Proposition 5.4.3 (Tonge, 1976)**.** *An injective Banach algebra is an operator algebra.*

Proof. Supposing that A is an injective Banach algebra we use the abbreviation

$$Y = \underbrace{A_1^* \times \cdots \times A_1^*}_{n \text{ copies}}$$

and write $(g_1, \dots, g_n) = \mathbf{g}$ for a typical element of Y. Let K to be the constant from the conclusions of Theorem 5.4.1, without loss of generality supposed to be no less than one. Thus for any $n \geq 1$ and $f \in A_1^*$ there is a complex measure $\mu = \mu_{n,f}$ on Y with $\|\mu_{n,f}\| \leq K^{n-1}$ and

$$f(a_1 \dots a_n) = \int_Y g_1(a_1) \dots g_n(a_n) d\mu(\mathbf{g}) \quad (a_1, \dots, a_n).$$

Then by standard results in measure theory (see, for example, [28, Th. 14.12 & 14.13]) there is a $|\mu|$-measurable function on Y, denoted $d\mu/d|\mu|$, satisfying

$$\left| \frac{d\mu}{d|\mu|}(\mathbf{g}) \right| = 1 \quad (\mathbf{g} \in Y) \tag{5.12}$$

and

$$\int_Y \phi \, d\mu = \int_Y \phi \frac{d\mu}{d|\mu|} d|\mu| \quad (\phi \in L^1(Y, |\mu|)).$$

We write $\mathscr{H} = \mathscr{H}_{n,f}$ for the space $L^2(Y, |\mu|)$, which is a Hilbert space with the inner product

$$\langle F, G\rangle = \int_Y F(\mathbf{g})\overline{G(\mathbf{g})}d|\mu|(\mathbf{g}) \quad (F, G \in \mathscr{H}).$$

For $a \in A$ and $i = 1, \ldots, n$ define operators $L_i(a)$ on $\mathscr{H}$ by

$$L_i(a)F(\mathbf{g}) = \begin{cases} K.g_i(a).F(\mathbf{g}) & i = 1, \ldots, n-1 \\ K.g_n(a).\dfrac{d\mu}{d|\mu|}(\mathbf{g})F(\mathbf{g}) & i = n \end{cases}$$

for $F \in \mathscr{H}$ and $\mathbf{g} \in Y$. Then for $i = 1, \ldots, n-1$ we have

$$\begin{aligned} \|L_i(a)F\|^2 &= \int_Y |L_iF(\mathbf{g})|^2 \, d|\mu|(\mathbf{g}) \\ &= K^2 \int_Y |g_i(a).F(\mathbf{g})|^2 \, d|\mu|(\mathbf{g}) \\ &\leq K^2 \|a\|^2 \int_Y |F(\mathbf{g})|^2 \, d|\mu|(\mathbf{g}) \\ &= K^2 \|a\|^2 \|F\|^2 \end{aligned}$$

and so $\|L_i(a)\|_{\mathrm{op}} \leq K \|a\|$. The corresponding inequality for $L_n(a)$ is derived similarly, but uses (5.12).

Now, for any $F \in \mathscr{H}$,

$$\begin{aligned} L_1(a_1) \ldots L_n(a_n)F(\mathbf{g}) &= L_1(a_1) \left(L_2(a_2) \ldots L_n(a_n)F\right)(\mathbf{g}) \\ &= K.g_1(a_1).\left(L_2(a_2) \ldots L_n(a_n)F\right)(\mathbf{g}) \\ &\vdots \\ &= K^{n-1}.g_1(a_1) \ldots g_{n-1}(a_{n-1}).L_n(a_n)F(\mathbf{g}) \\ &= K^n.g_1(a_1) \ldots g_n(a_n).\frac{d\mu}{d|\mu|}(\mathbf{g})F(\mathbf{g}). \end{aligned} \tag{5.13}$$

If $H \in \mathscr{H}$ is the function with constant value $K^{-n/2}$ on Y then

$$\|H\|^2 = \frac{\|\mu\|}{K^n} \leq \frac{1}{K}$$

by choice of K. Moreover from (5.13) we have

$$\begin{aligned} \langle L_1(a_1) \ldots L_n(a_n)H, H\rangle &= \int_Y L_1(a_1) \ldots L_n(a_n)H(\mathbf{g})\overline{H(\mathbf{g})} \, d|\mu|(\mathbf{g}) \\ &= \int_Y g_1(a_1) \ldots g_n(a_n).\frac{d\mu}{d|\mu|}(\mathbf{g}) \, d|\mu|(\mathbf{g}) \\ &= \int_Y g_1(a_1) \ldots g_n(a_n) \, d\mu(\mathbf{g}) \\ &= f(a_1 \ldots a_n) \end{aligned}$$

by choice of μ. We see that the hypotheses of Theorem 5.4.2 are satisfied, and so A is an operator algebra. $\square$

Recently C. Le Merdy has asked (P. G. Dixon, in a personal communication) whether the Banach algebra $\mathscr{B}(\mathscr{H})\hat{\otimes}\ell^1$ is an operator algebra when $\mathscr{H}$ is an infinite dimensional Hilbert space. The following shows that, unfortunately, Proposition 5.4.3 provides no assistance on this question.

Proposition 5.4.4. *The Banach algebra $A := \mathscr{B}(\mathscr{H})\hat{\otimes}\ell^1$ is not injective.*

Proof. Suppose that $K > 0$ is given. Since $\mathscr{B}(\mathscr{H})$ is not injective, we can find operators $T_1, \dots, T_n, S_1, \dots, S_n \in \mathscr{B}(\mathscr{H})$ satisfying

$$\left\| \sum_{i=1}^n T_i S_i \right\|_{\mathrm{op}} \geq K \quad \text{and} \quad \left\| \sum_{i=1}^n T_i \otimes S_i \right\|_{\epsilon} \leq 1.$$

With (e_i) denoting the natural basis elements of ℓ^1, set

$$v = \sum_{i=1}^n \sum_{j=1}^n T_i \otimes e_j \otimes S_i \otimes e_j \in A\hat{\otimes}A$$

so that

$$\begin{aligned} R_A(v) &= \sum_{i=1}^n \sum_{j=1}^n (T_i \otimes e_j)(S_i \otimes e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n T_i S_i \otimes e_j \\ &= \left(\sum_{i=1}^n T_i S_i \right) \otimes \left(\sum_{j=1}^n e_j \right) \end{aligned}$$

and so

$$\|R_A(v)\|_\pi = \left\| \sum_{i=1}^n T_i S_i \right\|_{\mathrm{op}} \left\| \sum_{j=1}^n e_j \right\|_1 \geq Kn.$$

Thus our proof is complete when we have shown that $\|v\|_\epsilon \leq n$.

To show this we write

$$X = \mathrm{span}\,\{T_1, \dots, T_n, S_1, \dots, S_n\} \quad \text{and} \quad Y = \mathrm{span}\,\{e_1, \dots, e_n\}$$

and note that $X\hat{\otimes}Y \overset{1}{\hookrightarrow} A$, directly from the definition of the projective tensor product. So by [11, Th. 4.3] we have

$$(X\hat{\otimes}Y)\check{\otimes}(X\hat{\otimes}Y) \overset{1}{\hookrightarrow} A\check{\otimes}A$$

and, with this identification of $(X\hat{\otimes}Y)\check{\otimes}(X\hat{\otimes}Y)$ with a finite-dimensional (and so closed) subspace of $A\check{\otimes}A$, we see that any bounded functional on $(X\hat{\otimes}Y)\check{\otimes}(X\hat{\otimes}Y)$ can be extended to a functional on $A\check{\otimes}A$ of the same norm. Thus

$$\begin{aligned} \|v\|_\epsilon &= \sup\left\{|\Lambda(v)| : \Lambda \in (A\check{\otimes}A)_1^*\right\} \\ &= \sup\left\{|\Gamma(v)| : \Gamma \in \left((X\hat{\otimes}Y)\check{\otimes}(X\hat{\otimes}Y)\right)_1^*\right\} \end{aligned}$$

and this is tractable when we use the identification

$$\left((X\hat{\otimes}Y)\check{\otimes}(X\hat{\otimes}Y)\right)^* \stackrel{1}{\cong} (X\hat{\otimes}Y)^*\hat{\otimes}(X\hat{\otimes}Y)^*$$

which follows from [11, Th. 6.4] since $X\hat{\otimes}Y$ is a finite-dimensional space. So suppose that $\Gamma \in (X\hat{\otimes}Y)^*\hat{\otimes}(X\hat{\otimes}Y)^*$ has $\|\Gamma\|_\pi \le 1$. Then, for each $\epsilon > 0$, we can find $F_k,\ G_k \in (X\hat{\otimes}Y)^*$ $(k = 1, \ldots, m)$ with

$$\Gamma = \sum_{k=1}^m F_k \otimes G_k \quad \text{and} \quad \sum_{k=1}^m \|F_k\| \, \|G_k\| \le 1 + \epsilon.$$

With these F_k and G_k fixed we define $\widetilde{F}_{k,j}, \widetilde{G}_{k,j} \in X^*$ by

$$\widetilde{F}_{k,j}(T) = F_k(T \otimes e_j), \quad \widetilde{G}_{k,j}(T) = G_k(T \otimes e_j) \quad (T \in X,\ k = 1, \ldots, n)$$

noting that

$$\left|\widetilde{F}_{k,j}(T)\right| \le \|F_k\| \, \|T \otimes e_j\|_\pi = \|F_k\| \, \|T\|_{\mathrm{op}}$$

so $\left\|\widetilde{F}_{k,j}\right\| \le \|F_k\|$ for all feasible k and j, a similar inequality holding for $\left\|\widetilde{G}_{k,j}\right\|$. Now

$$\begin{aligned} \Gamma(v) &= \sum_{k=1}^m F_k \otimes G_k \left(\sum_{i=1}^n \sum_{j=1}^n T_i \otimes e_j \otimes S_i \otimes e_j \right) \\ &= \sum_{j=1}^n \sum_{k=1}^m \sum_{i=1}^n F_k(T_i \otimes e_j) G_k(S_i \otimes e_j) \\ &= \sum_{j=1}^n \sum_{k=1}^m \sum_{i=1}^n \widetilde{F}_{k,j}(T_i) \widetilde{G}_{k,j}(S_i) \end{aligned}$$

so that

$$\begin{aligned}
|\Gamma(v)| &\leq \sum_{j=1}^{n}\sum_{k=1}^{m}\left|\sum_{i=1}^{n}\widetilde{F}_{k,j}(T_i)\widetilde{G}_{k,j}(S_i)\right| \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{m}\left\|\widetilde{F}_{k,j}\otimes\widetilde{G}_{k,j}\right\|_{\pi}\left\|\sum_{i=1}^{n}T_i\otimes S_i\right\|_{\epsilon} \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{m}\left\|\widetilde{F}_{k,j}\right\|\left\|\widetilde{G}_{k,j}\right\| \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{m}\|F_k\|\,\|G_k\| \\
&\leq n(1+\epsilon).
\end{aligned}$$

Thus, since ϵ is arbitrary, we have $|\Gamma(v)| \leq n$. Since this holds for each Γ we conclude that $\|v\|_{\epsilon} \leq n$, which completes the proof. ☐

5.5. A Generalisation of Q-algebras

In proving that a commutative injective Banach algebra is a Q-algebra [50, Th. 1], Varopoulos uses the following characterization of Q-algebras due to Craw (for a proof see [8, Prop. 50.5]). For a polynomial $p(X_1, \ldots, X_n)$ in m indeterminates, a Banach algebra A and $\delta > 0$ write

$$\|p\|_{A,\delta} = \sup\{\|p(a_1, \ldots, a_m)\| : \, a_i \in A, \, \|a_i\| \leq \delta, \, i = 1, \ldots, m\}.$$

Theorem 5.5.1 (Craw's Lemma)**.** *A commutative Banach algebra A is isomorphic to the quotient of a uniform algebra by a closed ideal (i.e. A is a Q-algebra) if and only if there are K, $\delta > 0$ such that*

$$\|p\|_{A,\delta} \leq K\,\|p\|_{\mathbb{C},1}$$

for all polynomials p. The same condition, but with $K = \delta = 1$ characterizes commutative Banach algebras isometric with the quotient of a uniform algebra by a closed ideal.

In [16] Dixon extends the methods of Craw to obtain a characterization of operator algebras. In this section we describe a class of Banach algebras (intermediate to operator algebras and Q-algebras) which have a similar characterization.

Definition 5.5.2. A Banach algebra A is a *PIQ-algebra* (*IPIQ-algebra*) if there is a C^*-algebra C satisfying a polynomial identity, a closed subalgebra $B \subseteq C$ and a closed ideal $I \lhd B$ such that A is isomorphic (isometric) to B/I.

By Gelfand-Naimark this definition reduces to that of a Q-algebra when the polynomial in question is that of commutativity: $p(X,Y) = XY - YX$. Thus PIQ-algebras can be seen as a natural generalisation of Q-algebras, and one asks if Varopoulos's methods can be applied to show that an injective Banach algebra satisfying a polynomial identity is necessarily a PIQ-algebra. The following version of Craw's result can be viewed as a step in this direction.

Proposition 5.5.3. *A Banach algebra A is a PIQ-algebra if and only if there are K, $\delta > 0$ and $n \in \mathbb{N}$ such that*

$$\|p\|_{A,\delta} \leq K \|p\|_{M_n(\mathbb{C}),1}$$

for all polynomials p. If we take $K = \delta = 1$ then the above condition characterizes IPIQ-algebras.

To prove this proposition we will need a brace of lemmas. The first is surely known and we give a proof only because we cannot provide a reference.

Lemma 5.5.4. *For each polynomial p and each closed ideal I of a Banach algebra B the inequality*

$$\|p\|_{B/I,1} \leq \|p\|_{B,1}$$

obtains.

Proof. As usual we write $[b]$ for the equivalence class containing $b \in B$. If $\epsilon > 0$ is given, then for any $[b_1], \ldots, [b_m] \in B/I$ with norms no greater than one there are $d_i \in I$ with $\|b_i + d_i\| \leq 1 + \epsilon$ for $i = 1, \ldots, m$. Then

$$p(b_1 + d_1, \ldots, b_m + d_m) = p(b_1, \ldots, b_m) + q$$

where $q \in I$, and so

$$\begin{aligned} \|p(b_1 + d_1, \ldots, b_m + d_m)\| &\geq \|[p(b_1 + d_1, \ldots, b_m + d_m)]\|_{B/I} \\ &= \|[p(b_1, \ldots, b_m) + q]\|_{B/I} \\ &= \|p([b_1], \ldots, [b_m])\|_{B/I} \\ &\geq \|p\|_{B/I,1} \end{aligned}$$

from which we conclude that

$$\|p\|_{B/I,1} \leq \|p\|_{B,1+\epsilon}\,. \tag{5.14}$$

To complete the proof we note that if $a_i \in B$ has $\|a_i\| \leq 1 + \epsilon$ then we can write $a_i = e_i + \epsilon f_i$ where $e_i,\ f_i \in B$ have norm no greater that one. Then for some $M \in \mathbb{N}$ and some polynomials $q_1, \ldots, q_M$ (depending only on p)

$$p(a_1, \ldots, a_m) = p(e_1, \ldots, e_m) + \sum_{i=1}^{M} \epsilon^i q_i(e_1, \ldots, e_m, f_1, \ldots, f_m)$$

and so

$$\|p(a_1,\ldots,a_m)\| \leq \|p\|_{B,1} + \sum_{i=1}^{M} \epsilon^i \|q_i\|_{B,1}\,.$$

Taking the supremum aver all such a_i we have

$$\|p\|_{B,1+\epsilon} \leq \|p\|_{B,1} + \sum_{i=1}^{M} \epsilon^i \|q_i\|_{B,1}\,.$$

which, in combination with (5.14) and letting $\epsilon \to 0$, gives the required inequality. □

The second lemma is proved in [40, Th. 5.1.2].

Lemma 5.5.5. *If C is a C^*-algebra then for each $a \in C$ there is an irreducible representation π of C (on some Hilbert space) such that $\|\pi(a)\| = \|a\|$.*

Proof of the Proposition. We prove the Proposition only in the *PIQ*-algebra case since the *IPIQ*-algebra case is the same, bar a change in some constants.

So suppose that A is a *PIQ*-algebra and, with the notation of Definition 5.5.2, suppose that ϕ is an isomorphism $A \to B/I$. Then for any polynomial p

$$\|p\|_{A,\|\phi\|^{-1}} \leq \left\|\phi^{-1}\right\| \|p\|_{B/I,1} \leq \left\|\phi^{-1}\right\| \|p\|_{B,1} \tag{5.15}$$

by Lemma 5.5.4. Now, since C is a C^*-algebra it is semisimple, and it is known that a semisimple algebra (not even necessarily normed) which satisfies a polynomial identity also satisfies a standard identity

$$S_{2n}(X_1,\ldots,X_{2n}) = \sum_{\sigma} \operatorname{sgn}(\sigma) X_{\sigma(1)} \ldots X_{\sigma(2n)}$$

where the summation is over all permutations on $1,\ldots,2n$ (see [44, Th. 1.6.42]). Then, by a previously mentioned result of Johnson [33, Prop. 6.1], each irreducible representation of C has dimension no greater than n. Thus if p is a polynomial and $b_1,\ldots,b_n \in B$ have norm no greater than one, then by Lemma 5.5.5 we can find some irreducible representation π of C, with dimension $k \leq n$ and

$$\begin{aligned}
\|p(b_1,\ldots,b_m)\| &= \|\pi\left(p(b_1,\ldots,b_m)\right)\| \\
&= \|p\left(\pi(b_1),\ldots,\pi(b_m)\right)\| \\
&\leq \|p\|_{M_k(\mathbb{C}),1} \\
&\leq \|p\|_{M_n(\mathbb{C}),1}\,.
\end{aligned}$$

Taking the supremum over all such b_i we obtain

$$\|p\|_{B,1} \leq \|p\|_{M_n(\mathbb{C}),1}$$

which, combined with (5.15), proves the proposition in one direction.

The proof in the opposite direction is the same as that of the main theorem of Dixon's paper [16]. There it is shown that if A is a Banach algebra, such that there are $K,\ \delta > 0$ with

$$\|p\|_{A,\delta} \leq \|p\|_{\mathscr{B}(\mathscr{H}),1}$$

for all polynomials p, then A is isomorphic to a subalgebra of the C^*-algebra $B(X, \mathscr{B}(\mathscr{H}))$ of bounded functions from some set X to $\mathscr{B}(\mathscr{H})$. (In fact this is shown in the case that $\mathscr{H}$ is infinite dimensional, but the proof makes no use of this assumption.) Thus one only needs observe that $B(X, \mathscr{B}(\mathscr{H}))$ satisfies the polynomials that are satisfied by $M_n(\mathbb{C})$ to complete the proof. □

We conclude by mentioning that Proposition 5.5.3 shows that, for each $n \in \mathbb{N}$, the class of Banach algebras A satisfying

$$\|p\|_{A,1} \leq \|p\|_{M_n(\mathbb{C}),1}$$

for all polynomials p, is a *variety* of Banach algebras in the sense of Dixon [15].

Appendix

A. Some Heuristic Diagrams

This appendix consists of a number of diagrams which illustrate different ways in which the quantity $\|a^n\|^{1/n}$ might converge to the spectral radius of a in a Banach algebra. This illustration is qualitative rather than quantitative, for quite obvious reasons, and this renders our diagrams sufficiently heuristic to exclude them from the main text.

Our method of illustration is to draw a 3-dimensional graph with elements a of the algebra A represented on one horizontal axis, the positive integers $n \in \mathbb{N}$ on the other and $\|a^n\|^{1/n}$ on the vertical axis. Of course a certain suspension of disbelief is required to imagine an infinite dimensional complex Banach algebra represented by an interval, and to obtain a finite diagram we must scale the integers in some fashion. We go a little further, in the pursuit of legibility, in drawing as if n were a continuous variable.

The first three figures illustrate the qualitative differences between bounded index and topologically bounded index in a Banach algebra and are self-explanatory. In the final figure is a little more quantitative, with the n axis running from 1 to 30 and the A axis representing the 10×10 matrices a_λ

$$a_\lambda = \begin{bmatrix} \lambda & 1-\lambda & & \\ & \lambda & \ddots & \\ & & \ddots & 1-\lambda \\ & & & \lambda \end{bmatrix}$$

for $\lambda \in [0, 1]$. We choose these matrices since they constitute a path in $M_{10}(\mathbb{C})$ starting with a nilpotent, passing through (a scalar multiple of) the matrix used to obtain a lower bound on $V_{M_n(\mathbb{C})}$ in (3.8), and ending with the unit.

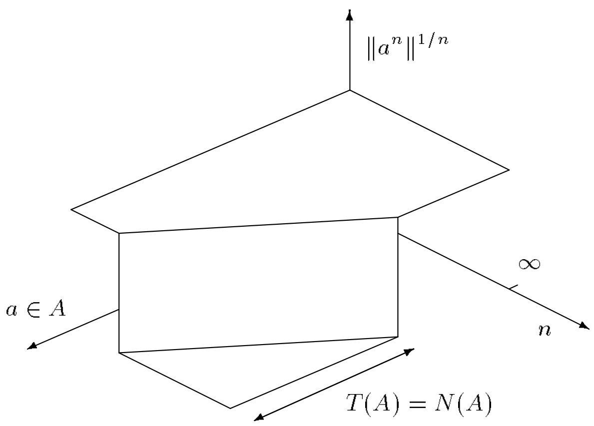

Figure A.1.: Neither bounded index nor topologically bounded index

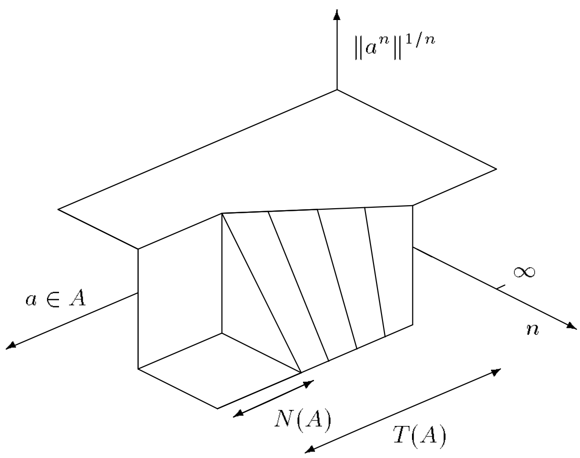

Figure A.2.: Bounded index but not topologically bounded index

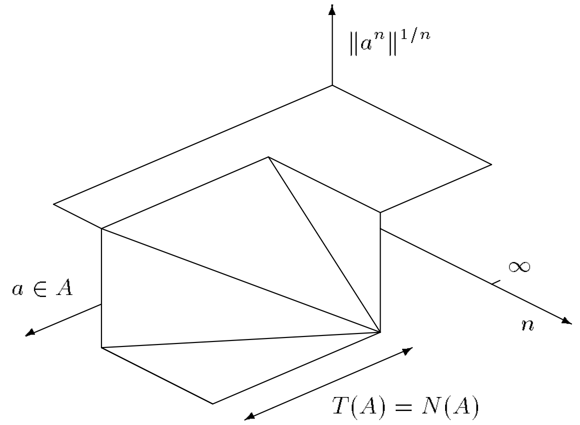

Figure A.3.: Topologically bounded index but not bounded index

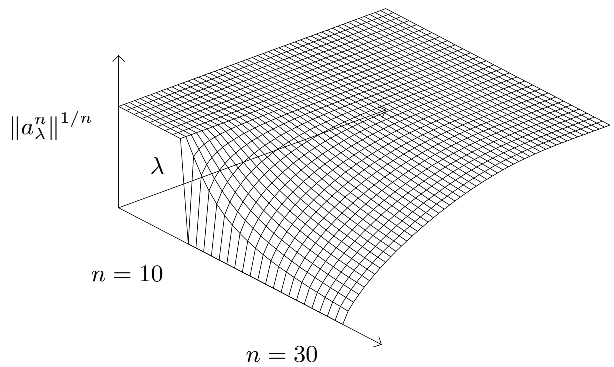

Figure A.4.: A 'real' example. Here a_λ is the 10×10 matrix with λ on the diagonal and $1 - \lambda$ on the first superdiagonal, where $0 \leq \lambda \leq 1$.

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