Injective semigroup-algebras*

J. J. Green

June 15, 2002

Abstract

Semigroups S for which the Banach algebra $\ell^1(S)$ is injective are investigated and an application to the work of O. Yu. Aristov is described.

1 Introduction

Injective Banach algebras were introduced by Varopoulos in [12] and have continued to attract investigation some 25 years later. In this note make some progress towards a structural description of the semigroups S for which the Banach algebra $\ell^1(S)$ is injective.

To introduce our notation suppose that A and B are Banach algebras. We write $A \otimes B$ for the algebraic tensor product over $\mathbf{C}$, and $A \otimes_\epsilon B$ for $A \otimes B$ equipped with (but not completed in) the injective tensor norm

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_\epsilon := \sup \left\{ \left| \sum_{i=1}^n f(a_i) g(b_i) \right| : f \in A_1^*,\, g \in B_1^* \right\}.$$

Tensor products and tensor norms are given a detailed treatment in [5], while [3, §42] provides an introduction. Following Varopoulos we will say that a Banach algebras A is *injective* if the mapping

$$\begin{aligned} R_A : A \otimes_\epsilon B &\longrightarrow A \\ \sum_{i=1}^n a_i \otimes b_i &\longmapsto \sum_{i=1}^n a_i b_i \end{aligned}$$

often called the *product morphism*, is bounded.

If S is a semigroup we write $\mathbf{C}[S]$ for the algebra of formal sums

$$x = \sum_{s \in S} \xi_s s \tag{1}$$

for which only finitely many of the $\xi_s \in \mathbf{C}$ are non-zero. When equipped with the ℓ^1 norm

$$\|x\|_1 := \sum_{s \in S} |\xi_s|$$

$\mathbf{C}[S]$ is a normed algebra whose completion is the ℓ^1 *semigroup-algebra* universally denoted $\ell^1(S)$. We will always assume that our semigroups are countable and we use the notational convention that a semigroup S has an unspecified but fixed enumeration of its elements i.e. that $S = \{s_i : i \in \mathbf{N}\}$.

*1991 Mathematics subject classification 46H05

2 Necessary conditions

It is well-known that an injective Banach algebra is an operator algebra [10, Th. 4.2.26] and so Arens regular [4]. Thus necessary conditions for a semigroup S to have $\ell^1(S)$ injective follow immediately from the characterization of the semigroups S for which $\ell^1(S)$ is Arens regular [11], [13] [2]. Indeed the title of [11] indicates that the Arens regularity of $\ell^1(S)$ places strong restrictions on the structure of S, hence the injectivity of $\ell^1(S)$ more so.

The following lemma enables us to utilise the results above but at the same time to exploit the stronger hypothesis of injectivity.

Lemma 2.1. *Suppose that S and T are semigroups and that $\ell^1(T)$ is not injective. Suppose further that there are finite subsets T_1, $T_2, \ldots$ with*

$$T_1 \subseteq T_2 \subseteq \cdots \subseteq T$$

whose union is T and, if $m = m(n)$ denotes the smallest integer such that $T_n^2 \subseteq T_m$, that there are maps

$$\psi_n : T_{m(n)} \longrightarrow S \quad (n \in \mathbf{N})$$

with

$$\psi_n(a)\psi_n(b) = \psi_n(ab) \quad (a,\, b \in T_n,\ n \in \mathbf{N}).$$

Then $\ell^1(S)$ is not injective.

Proof. If $K > 0$ is given then, since $\ell^1(T)$ is not injective, there is some $u \in \ell^1(T) \otimes_\epsilon \ell^1(T)$ with $\|u\|_\epsilon \leq 1$ and $\left\|R_{\ell^1(T)}(u)\right\|_1 \geq K$. Indeed, by a density argument, we assume that u has a representation as a finite sum

$$u = \sum_{i,j} \xi_{i,j} a_i \otimes b_j \quad (a_i, b_j \in T)$$

and take n to be a number such that a_i and b_j are in T_n whenever $\xi_{i,j} \in \mathbf{C}$ is non-zero. The map ψ_n has an obvious linearisation which we also denote ψ_n when we define

$$v = \sum_{i,j} \xi_{i,j} \psi_n(a_i) \otimes \psi_n(b_j) \in \ell^1(S) \otimes_\epsilon \ell^1(S).$$

Then we have

$$\begin{aligned} \|v\|_\epsilon &= \sup\left\{ \left| \sum_{i,j} \xi_{i,j} f\left(\psi_n(a_i)\right) g\left(\psi_n(b_j)\right) \right| : f, g \in \left(\ell^1(S)\right)_1^* \right\} \\ &\leq \sup\left\{ \left| \sum_{i,j} \xi_{i,j} F(a_i) G(b_j) \right| : F,\, G \in \left(\ell^1(T)\right)_1^* \right\} \\ &= \|u\|_\epsilon \end{aligned} \tag{2}$$

since $f \circ \psi_n$ and $g \circ \psi_n$ are linear functionals on $\ell^1(T_m)$ of norm no greater than one, and so may be extended to such on $\ell^1(T)$ by the Hahn-Banach theorem. Then

$$R_{\ell^1(S)}(v) = \sum_{i,j} \xi_{i,j} \psi_n(a_i) \psi_n(b_j) = \psi_n\left(R_{\ell^1(T)}(u)\right)$$

so that

$$\left\|R_{\ell^1(S)}(v)\right\|_1 = \left\|\psi_n\left(R_{\ell^1(T)}(u)\right)\right\|_1 \geq K \|u\|_\epsilon \geq \|v\|_\epsilon$$

by (2), which completes the proof. □

Our first application of the lemma is to show that semigroups S with $\ell^1(S)$ injective are "uniformly periodic".

PROPOSITION 2.2. *If S is a semigroup with $\ell^1(S)$ is injective then there is a number $N \in \mathbf{N}$ such that*

$$\operatorname{card}\{s^n : n \in \mathbf{N}\} \leq N \quad (s \in S).$$

In particular such a semigroup is periodic.

Proof. If there is no such N then for each $n \in \mathbf{N}$ we can find some $s \in S$ such that $s, s^2, \ldots, s^{2n}$ are distinct. Writing $T_n = \{1, 2, \ldots, n\}$ (considered as a subset of the semigroup of $\mathbf{N}$ with addition as product) and defining

$$\begin{array}{rcl} \psi_n : T_{2n} & \longrightarrow & S \\ i & \longmapsto & s^i \end{array}$$

we see that the conditions of the lemma are met once we have shown that the semigroup $T = (\mathbf{N}, +)$ has a semigroup algebra which is not injective. But it is not even Arens regular, as is shown by a straightforward application of [2, Th. 2.7]. □

The hypothesis of injectivity in Proposition 2.2 cannot be weakened to that of Arens regularity. To see this we observe the following fact whose proof, again, is a consequence of [2, Th. 2.7].

PROPOSITION 2.3. *Let S be a semigroup with zero θ such that for each $s \in S$ there are only finitely many $r \in S$ such that $rs \neq \theta$ and only finitely many $t \in S$ such that $st \neq \theta$. Then $\ell^1(S)$ is Arens regular.*

The conditions of Proposition 2.3 are met by the semigroup S which is the zero direct product [7, Ch 3, Sect. 3] of a sequence of cyclic groups of increasing order. So we find a semigroup that clearly does not satisfy the conditions of Proposition 2.2, but whose semigroup algebra is not Arens regular.

The second application of Lemma 2.1 concerns the set $E(S)$ of idempotents in a semigroup S. Let $\leq$ denote the partial order on $E(S)$ defined by

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

PROPOSITION 2.4. *Let S be a semigroup such that $\ell^1(S)$ is injective. Then there is a number $N \in \mathbf{N}$ such that no chain of idempotents in $E(S)$ exceeds N in length.*

Proof. If there is no such N then, for each $n \in \mathbf{N}$, we can find some chain of n idempotents, say $e_n \leq e_{n-1} \leq \cdots \leq e_1$. Writing $T_n = \{1, \ldots, n\}$ (considered as a subset of the semigroup of $\mathbf{N}$ with the max product) and defining

$$\begin{array}{rcl} \psi_n : T_n & \longrightarrow & S \\ i & \longmapsto & e_{n-i+1} \end{array}$$

we see that the conditions of Lemma 2.1 are met once we show that the semigroup $(\mathbf{N}, \max)$ has a semigroup algebra which is not injective. Again [2, Th. 2.7] shows that it is not even Arens regular. □

One may show that Arens regularity cannot replace injectivity as the hypothesis of Proposition 2.4. The method is similar to the above — consider the semigroup which is a zero direct sum of a sequence of chains of increasing order. Notice, however, that the Arens regularity of $\ell^1(S)$ implies that the chains of idempotents in S must at least be finite, else S has a sub-semigroup isomorphic to $(\mathbf{N}, \max)$.

The contrast between the associations of Arens regularity with finiteness and injectivity with uniform boundedness seems a theme of subject and is maintained in the next section.

3 Sufficient conditions for semigroups with zero

For semigroups S with zero there are some conditions that force the injectivity of $\ell^1(S)$; conditions which prescribe the *sparsity* of non-zero products in S. Our approach to these is via a well known algebraic construction.

If S is a semigroup with zero θ then we will write $\mathbf{C}_r[S]$ for the *reduced semigroup-algebra* of S; the linear algebra $\mathbf{C}[S]/\mathbf{C}[\{\theta\}]$, and denote by $\ell^1_r(S)$ the completion of $\mathbf{C}_r(S)$ in the ℓ^1 norm

$$\left\| \sum_{s\in S\setminus\{\theta\}} \xi_s s \right\|_1 := \sum_{s\in S\setminus\{\theta\}} |\xi_s| .$$

Our interest in such algebras lies in the following fact whose proof is, but for a change in notation, essentially the argument used by Varopoulos in [12] (and attributed there to S. Kaijser) to show that ℓ^1 is injective.

Lemma 3.1 (Varopoulos 1972). *Let $S = \{\theta, e_1, e_2, \ldots\}$ be a countable semigroup with zero θ and suppose that*

$$u = \sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j \in \ell^1_r(S) \otimes_\epsilon \ell^1_r(S).$$

Then for any permutation σ on $\{1, \ldots, m\}$ the inequality

$$\sum_{i=1}^{m} \left|\xi_{i,\sigma(i)}\right| \le \|u\|_\epsilon$$

obtains.

Proposition 3.2. *Let S be a countable semigroup with zero θ. Suppose that there is some $K \in \mathbf{N}$ such that for each non-zero $s \in S$ there are at most K elements $t \in S$ with $st \neq \theta$ and at most K elements $r \in S$ with $rs \neq \theta$. Then $\ell^1_r(S)$ is injective and $\left\|R_{\ell^1_r(S)}\right\| \le K$.*

Proof. We write $S = \{\theta, e_1, e_2, \ldots\}$ and suppose that $u \in \ell^1_r(S) \otimes_\epsilon \ell^1_r(S)$ is of the form

$$u = \sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j. \tag{3}$$

We set $M = \max\{n : e_i e_j = e_n \text{ for some } i, j = 1, 2, \ldots, m\}$ so that

$$R_{\ell^1_r(S)}(u) = \sum_{i,j=1}^{m} \xi_{i,j} e_i e_j = \sum_{k=1}^{M} \left(\sum_{e_i e_j = e_k} \xi_{i,j} \right) e_k$$

from which we obtain the inequality

$$\left\|R_{\ell_r^1(S)}(u)\right\| \leq \sum_{\substack{1\leq i,j\leq m \\ e_ie_j\neq\theta}} |\xi_{i,j}| \,. \tag{4}$$

Setting

$$\lambda_{i,j} = \begin{cases} 0 & \text{if } e_ie_j = \theta \text{ or} \\ |\xi_{i,j}| & \text{otherwise} \end{cases}$$

we see that the right-hand side of (4) is the summation over the elements of the $m \times m$ matrix $\Lambda = [\lambda_{i,j}]$, a matrix which has at most K non-zero elements in each row and in each column. Such a matrix can be written as the sum of exactly K matrices with at most one non-zero element of Λ in each row and in each column (this is shown in Mirsky's book [9, Th. 11.1.6]) and so the right-hand side of (4) is the sum of exactly K sums of the form

$$\sum_{i=1}^{m} \left|\xi_{i,\sigma(i)}\right| .$$

Hence, applying Lemma 3.1, we find that

$$\left\|R_{\ell_r^1(S)}(u)\right\| \leq K\|u\|_\epsilon$$

for all u of the form (3). The result now follows from the fact that such elements are dense in $\ell_r^1(S) \otimes_\epsilon \ell_r^1(S)$. □

Notice that Proposition 3.2 applied to the semigroup $S = \{\theta, e_1, e_2, \ldots\}$ with product

$$e_ie_j = \begin{cases} e_i & \text{if } i = j, \text{ or} \\ \theta & \text{otherwise} \end{cases}$$

shows that $\ell_r^1(S)$, which is clearly isomorphic with ℓ^1, is injective. Thus we recover the result and implicit bound described in Varopoulos *ibid*.

We can apply Proposition 3.2 to the subject of this article by use of the following theorem.

THEOREM 3.3. *Let S be a countable semigroup with zero θ and such that $\ell_r^1(S)$ is injective. Then $\ell^1(S)$ is injective and*

$$\left\|R_{\ell^1(S)}\right\| \leq 6\left\|R_{\ell_r^1(S)}\right\| + 1.$$

Proof. We will write $S = \{e_0, e_1, \ldots\}$, where $e_0 = \theta$, for simplicity of notation. If

$$u = \sum_{i,j=0}^{m} \xi_{i,j} e_i \otimes e_j \in \ell^1(S) \otimes_\epsilon \ell^1(S) \tag{5}$$

then

$$\left|\sum_{i,j=0}^{m} \xi_{i,j}\right| \leq \|u\|_\epsilon$$

and since

$$\sum_{e_ie_j=e_0} \xi_{i,j} = \sum_{k=0}^{\infty}\left(\sum_{e_ie_j=e_k} \xi_{i,j}\right) - \sum_{k=1}^{\infty}\left(\sum_{e_ie_j=e_k} \xi_{i,j}\right)$$

we find that

$$\begin{aligned}\left|\sum_{e_ie_j=e_0}\xi_{i,j}\right| &\le \left|\sum_{i,j=0}^{m}\xi_{i,j}\right| + \sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\xi_{i,j}\right| \\ &\le \|u\|_\epsilon + \sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\xi_{i,j}\right|\end{aligned}$$

which gives that

$$\left\|R_{\ell^1(S)}(u)\right\|_1 = \sum_{k=0}^{\infty}\left|\sum_{e_ie_j=e_k}\xi_{i,j}\right| \le \|u\|_\epsilon + 2\sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\xi_{i,j}\right|. \tag{6}$$

Now

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\sum_{e_ie_j=e_k}\xi_{i,j}\right| &= \left\|R_{\ell^1_r(S)}\left(\sum_{i,j=1}^{m}\xi_{i,j}e_i\otimes e_j\right)\right\|_1 \\ &\le \left\|R_{\ell^1_r(S)}\right\|\left\|\sum_{i,j=1}^{m}\xi_{i,j}e_i\otimes e_j\right\|_{\ell^1_r(S)\otimes_\epsilon\ell^1_r(S)} \\ &= \left\|R_{\ell^1_r(S)}\right\|\left\|\sum_{i,j=1}^{m}\xi_{i,j}e_i\otimes e_j\right\|_{\ell^1(S)\otimes_\epsilon\ell^1(S)},\end{aligned} \tag{7}$$

since injective tensor products preserve subspaces [5, §4.3], and since

$$\sum_{i,j=1}^{m}\xi_{i,j}e_i\otimes e_j = u - \left(\sum_{i=0}^{m}\xi_{i,0}e_i\right)\otimes e_0 - e_0\otimes\left(\sum_{j=1}^{m}\xi_{0,j}e_j\right)$$

we have

$$\begin{aligned}\left\|\sum_{i,j=1}^{m}\xi_{i,j}e_i\otimes e_j\right\|_\epsilon &\le \|u\|_\epsilon + \|e_0\|_1\left(\left\|\sum_{i=0}^{m}\xi_{i,0}e_i\right\|_1 + \left\|\sum_{j=1}^{m}\xi_{0,j}e_j\right\|_1\right) \\ &= \|u\|_\epsilon + \sum_{i=0}^{m}|\xi_{i,0}| + \sum_{j=1}^{m}|\xi_{0,j}| \\ &\le 3\|u\|_\epsilon.\end{aligned} \tag{8}$$

Combining the inequalities (6), (7) and (8) now gives the bound

$$\left\|R_{\ell^1(S)}(u)\right\|_1 \le 6\left\|R_{\ell^1_r(S)}(u)\right\|_1 + \|u\|_1$$

for elements u of the form (5). This bound extends to the closure and so proves the theorem. □

Corollary 3.4. *Let S be a countable semigroup with zero θ. Suppose that there is some $K\in\mathbf{N}$ such that for each non-zero $s\in S$ there are at most K elements $t\in S$ with $st\ne\theta$ and at most K elements $r\in S$ with $rs\ne\theta$. Then $\ell^1(S)$ is injective and $\left\|R_{\ell^1(S)}\right\|\le 6K+1$.*

We remark that the above results do not provide a characterisation of the semigroups with zero such that $\ell^1(S)$ is injective. Consider the semigroup $S = \{\theta, e_1, e_2, \ldots\}$ with product $e_i e_j = e_i$ $(i, j \in \mathbf{N})$. Clearly S satisfies the conclusions of Propositions 2.2 and 2.4, while not the hypotheses of Corollary 3.4.

To conclude this section we invite the reader to compare Corollary 3.4 with Proposition 2.3.

4 The weighted case and an application

Some of what is described above can be extended to cover the weighted case: if S is a semigroup with zero θ say that $\omega : S\setminus\{\theta\} \to (0, \infty)$ is an *algebra weight* if

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t, st \in S\setminus\{\theta\}).$$

The weighted reduced semigroup algebra $\ell^1_r(S, \omega)$ is then defined analogously to $\ell^1_r(S)$; the completion of $\mathbf{C}_r[S]$ with respect to the norm

$$\left\| \sum_{s \in S\setminus\{\theta\}} \xi_s s \right\|_\omega := \sum_{s \in S\setminus\{\theta\}} |\xi_s| \, \omega(s).$$

In particular the following version of Varopoulos's Lemma holds, the proof again being an increment on that in [12].

LEMMA 4.1. *Let $S = \{\theta, e_1, e_2, \ldots\}$ be a countable semigroup with zero θ, ω an algebra weight and suppose that*

$$u = \sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j \in \ell^1_r(S) \otimes_\epsilon \ell^1_r(S).$$

Then for any permutation σ on $\{1, \ldots, m\}$ the inequality

$$\sum_{i=1}^{m} \left| \xi_{i,\sigma(i)} \right| \omega(e_i)\omega(e_{\sigma(i)}) \leq \|u\|_\epsilon$$

obtains.

The point of passing to the weighted case is that a sufficiently rapid rate of decrease in the weight can play the role that finiteness does in the unweighted case.

Let $e_{i,j}$ denote the infinite matrix with one as the i, j-th entry and zeros elsewhere, and θ the infinite matrix of zeros. Then with the usual matrix multiplication the set

$$S := \{e_{i,j} : 1 \leq i < j\} \cup \{\theta\}$$

is a semigroup with zero. Define a weight ω on $S\setminus\{\theta\}$ by

$$\omega(i, j) = 2^{-(j-i)^2} \quad (1 \leq i < j).$$

To see that this is an algebra-weight note that

$$\begin{aligned} \omega(i, j)\omega(j, k) &= 2^{-(j-i)^2-(k-j)^2} \\ &= 2^{2(j-i)(k-j)}\omega(i, k) \end{aligned}$$

and so, by a short calculation,

$$\begin{aligned}\omega(i,k) &\leq 2^{-2(j-i)(k-j)}\omega(i,j)\omega(j,k)\\ &\leq 2^{-2(k-i-1)}\omega(i,j)\omega(j,k).\end{aligned}$$

PROPOSITION 4.2. *With S and ω defined as above, the Banach algebra $A = \ell^1_r(S,\omega)$ is injective.*

Proof. Suppose that $u \in A \otimes_\epsilon A$ is of the form

$$u = \sum_{i<j,\, k<l} \xi_{i,j,k,l} e_{i,j} \otimes e_{k,l}$$

where only finitely many of the $\xi_{i,j,k,l}$ are non-zero. Then

$$R_A(u) = \sum_{i<j<l} \xi_{i,j,j,l} e_{i,l} = \sum_{m=2}^{\infty} \sum_{i<j<i+m} \xi_{i,j,j,i+m} e_{i,i+m}$$

so that

$$\begin{aligned}\|R_A(u)\|_\omega &= \sum_{m=2}^{\infty} \sum_{i<j<i+m} |\xi_{i,j,j,i+m}|\, \omega(i,i+m)\\ &\leq \sum_{m=2}^{\infty} \sum_{i<j<i+m} |\xi_{i,j,j,i+m}|\, 2^{-2(m-1)}\omega(i,j)\omega(j,i+m)\\ &= \sum_{m=2}^{\infty} 2^{-2(m-1)} \sum_{i<j<i+m} |\xi_{i,j,j,i+m}|\, \omega(i,j)\omega(j,i+m). \qquad (9)\end{aligned}$$

Now, with m fixed, for each pair (i,j) there is exactly one pair (k,l) such that $\xi_{i,j,k,l}$ occurs in the inner sum of (9). Thus, by a suitable relabelling of the semigroup elements $e_{i,j}$, we can apply Lemma 4.1 to obtain

$$\sum_{i<j<i+m} |\xi_{i,j,j,i+m}|\, \omega(i,j)\omega(j,i+m) \leq \|u\|_\epsilon \quad (m = 2, 3, \ldots)$$

and so from (9)

$$R_A(u) \leq \sum_{m=2}^{\infty} 2^{-2(m-1)} \|u\|_\epsilon = \frac{1}{3}\|u\|_\epsilon.$$

The result now follows since elements of the form u (i.e. those with finite support) are dense in $A \otimes_\epsilon A$. □

We find the injectivity of this example to be of interest for the following reason. In [1] Aristov shows that a C^*-algebra is injective if and only if it is subhomogeneous, i.e. if there is some uniform bound on the dimensions of its continuous irreducible representations. It is well known that a semisimple Banach algebra is subhomogeneous if and only if it satisfies a polynomial identity [8, Prop. 6.1] so it is natural to ask whether these three properties coincide for Banach algebras more general than C^*-algebras. That there are commutative semisimple Banach algebras which are not Arens regular (for example $\ell^1(\mathbf{Z})$) gives a negative answer in one direction while the above Proposition gives a partial negative answer in the opposite direction once we note that A does not satisfy a polynomial identity. If an algebra (not even necessarily normed) satisfies a polynomial identity then it satisfies a homogeneous multilinear identity of no greater degree [6, Lemma 6.2.4], so it suffices to show that A does not satisfy an identity of the form

$$p(X_1, \ldots, X_n) := X_1 \ldots X_n + \sum_{\sigma \neq 1} \lambda_\sigma X_{\sigma(1)} \ldots X_{\sigma(n)}$$

where the summation is over all non-trivial permutations on $\{1, \ldots, n\}$. But this is obvious since A contains half of "Kaplansky's staircase"

$$p(e_{1,2}, e_{2,3}, \ldots, e_{n,n+1}) = e_{1,n+2} \neq 0.$$

We conclude by mentioning that A is a radical Banach algebra and so trivially subhomogeneous. Thus it does provide an answer to the more difficult question as to whether there is a *semisimple* injective Banach algebra that does not satisfy a polynomial identity, or equivalently is not subhomogeneous.

References

[1] O. Yu. Aristov, *Characterization of strict C^*-algebras*, Studia Math. 112 (1994) pp. 51–58.

[2] J. W. Baker and A. Rejali, *On the Arens regularity of weighted convolution algebras*, J. London Math. Soc. (2) 40 (1989) pp. 535–546.

[3] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin, 1973.

[4] P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. 11 (1961) pp. 847 – 870.

[5] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam, 1993.

[6] I. N. Herstein, *Non-commutative Rings*, Mathematical Association of America, Washington, 1968.

[7] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[8] B. E. Johnson, *Near inclusions for subhomogeneous C^*-algebras*, Proc. London Math. Soc. (3) 68 (1994) pp. 399–422.

[9] L. Mirsky, *Transversal Theory*, Academic Press, London, 1971.

[10] T. W. Palmer, *Banach Algebras and the General Theory of *-algebras*, Cambridge University Press, Cambridge, 1994.

[11] J. S. Pym, *Convolution measure algebras are not usually Arens-regular*, Quart. J. Math. Oxford Ser. (2) 25 (1974) pp. 235–240.

[12] N. Th. Varopoulos, *Some remarks on Q-algebras*, Ann. Inst. Fourier (Grenoble) 22 (1972) pp. 1–11.

[13] N. J. Young, *Semigroup algebras having regular multiplication*, Studia Math. 47 (1973) pp. 191–196.

J. J. Green
c/o 22 Wansfell Rd.,
Grimesthorpe,
Sheffield, UK.
j.j.green@shef.ac.uk