The Bulletin of Symbolic Logic
Volume 21, Number 1, March 2015

THE CONVENIENCE OF THE TYPESETTER; NOTATION AND TYPOGRAPHY IN FREGE'S *GRUNDGESETZE DER ARITHMETIK*

J. J. GREEN, MARCUS ROSSBERG, AND PHILIP A. EBERT

Abstract. We discuss the typography of the notation used by Gottlob Frege in his *Grundgesetze der Arithmetik*.

§1. Background to the *Grundgesetze der Arithmetik*. *Grundgesetze der Arithmetik* was to have been the pinnacle of Gottlob Frege's life's work — a rigorous demonstration of how the fundamental laws of classical pure mathematics of the natural and real numbers can be derived from principles which, in Frege's view, were purely logical. His logical system, called *Begriffsschrift*, i.e., "concept-script", was first introduced in 1879 in his book with this title [19]. It includes the first occurrence in formal logic of quantifiers,[1] with which multiple and embedded generality could be expressed — no earlier logical system was capable of this. It also offers the first formulation of a logical system that contains relations rather than merely monadic predicates. In addition, Frege here presents his celebrated definition of the ancestral of a relation. Taken together, these developments made logic expressively adequate for mathematics for the first time in history [13, pp. xxxv–xxxvi]. *Begriffsschrift* is thus widely acknowledged as the greatest advance in logic since Aristotle — as W. V. Quine put it [43, p. vii]:

> Logic is an old subject, and since 1879 it has been a great one.

In 1884 Frege published the book *Die Grundlagen der Arithmetik* [20] in which he formulates and argues for logicism — the idea that arithmetic (and analysis) is reducible to logic. The principal aim of *Grundlagen* is to provide *philosophical* arguments for logicism, but Frege also offers proof-sketches of how Peano's axioms for arithmetic can be derived from entirely logical principles, based on an explicit definition of "cardinal number" and taking extensions of concepts as primitive. It was to be the task of his magnum opus, *Grundgesetze der Arithmetik* [21, 23], to show conclusively the purely logical nature of mathematics by presenting gapless proofs of the axioms of arithmetic and real analysis in his formal system, using only explicit definitions

Received October 21, 2014.
2000 *Mathematics Subject Classification.* 03A05, 01A55, 03Bxx, 00A30.
Key words and phrases. Gottlob Frege, logicism, logic, notation, typography.

[1] It is worthwhile mentioning that Frege's treatment includes higher-order quantification.

1079-8986/15/2101-0002
DOI:10.1017/bsl.2015.5

and six principles that he regarded to be basic laws of logic. These principles included the infamous Basic Law V (BLV) which governs the identity of value-ranges — more or less what we would consider graphs of functions today: BLV states that the value-ranges of functions f and g are identical if and only if f and g have the same values for all arguments. A special case of value-ranges are extensions: extensions are value-ranges of concepts, as Frege takes concepts to be functions from objects to truth-values. Accordingly, for the special case of extensions, BLV specifies that the extensions of concepts F and G are identical if and only if the same objects fall under F and G. As Frege points out in *Grundgesetze* (vol. II, §147), extensions are essentially what others call "classes", and he proceeds to use the term "class" instead of "extension of a concept" (vol. II, §161) throughout part III of *Grundgesetze*. By BLV, value-ranges (and extensions) are extensional, but BLV also entails something like an unrestricted ("naïve") comprehension principle for value-ranges (and extensions), which spells trouble.[2]

The first volume of *Grundgesetze* appeared in 1893 focusing on natural-number arithmetic, the second in 1903 containing first a philosophical discussion of attempts by other mathematicians to provide a foundation for real numbers, followed by the beginnings of Frege's logicist treatment of the real numbers.

A now famous letter he received in 1902 from Bertrand Russell [47], while the second volume of *Grundgesetze* was already in press (vol. II, p. 253),[3] made Frege realise that BLV leads to a contradiction. Frege delayed the publication of volume II of *Grundgesetze* by half a year, trying to solve the problem. In the end, he offered a fix in an afterword he appended to the second volume, concluding

> This question may be viewed as the fundamental problem of arithmetic: how are we to apprehend logical objects, in particular, the numbers? What justifies us to acknowledge numbers as objects? Even if this problem is not solved to the extent that I thought it was when composing this volume, I do not doubt that the path to the solution is found.

This "solution", however, later turned out to be unworkable as well: the revised Basic Law V (BLV$'$) is satisfiable only on a one-element domain.[4] Frege at some point must have realised that BLV$'$ would not do the work he envisioned. He never returned to the formal derivation of the basic laws of

[2] See e.g. Heck [34, ch. 1] for a presentation in modern notation.

[3] Implicit references of volume, section and page are understood to be to those of *Grundgesetze* [21, 23], or to the English translation [28] (which uses the same sectioning and pagination).

[4] The triviality of domains satisfying Frege's BLV$'$ was demonstrated by Stanisław Leśniewski, reported by Sobociński [49, §IV] and popularised by Quine [44]. In fact, BLV$'$ is inconsistent with Frege's stipulations regarding the truth-values in *Grundgesetze*, vol. I, §10, which entail that there are at least two objects (the True and the False). The issue whether these stipulations should be considered part of the formal system of *Grundgesetze* is contentious. See Dummett [12], Landini [39], Heck [34, ch. 4], Cook [10].

arithmetic in print. The projected third volume of *Grundgesetze*, which was to contain the definition of the real (and perhaps complex) numbers, and the derivation of the axioms of real (and perhaps complex) analysis, never appeared. If drafts existed in handwritten form, they were destroyed together with the rest of Frege's *Nachlass* during the 1945 bombing of Münster, Germany, where the documents were held in the university library.[5]

§2. The logical system of *Grundgesetze*. The logical system of *Grundgesetze* comprises the following primitives:[6]

Judgement-stroke: $\vdash$

Frege's theorem sign, which inspired the modern turnstile, $\vdash$.

Definition-stroke: $\Vdash$

Placed to the left of definitional equations, in place of the judgement-stroke (see for example Figure 1).

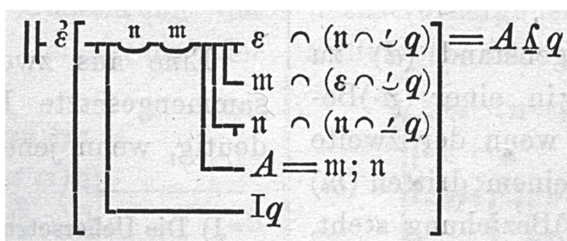

FIGURE 1. Concept-script of vol. I, §158.

Horizontal: —

Essentially, a truth-function that takes the True to the True, and everything else (the False as well as objects that are not truth-values) to the False.

Negation-stroke: $\top$

Frege's negation sign, a near analogue[7] of the truth-function $\neg$.

Conditional-stroke: $\begin{array}{l}\top\,\Gamma\\ \llcorner\,\Delta\end{array}$

Frege's conditional; taken in modern notation, the "direction" of the conditional is: $\Delta \supset \Gamma$.

---

[5]But see Wehmeier and Schmidt am Busch [57].

[6]Giving a rigorous introduction to Frege's system goes beyond the scope of this article. For a detailed explanation of Frege's logical system of *Grundgesetze* see Landini [40], Heck [34], and Cook [9]. When below we speak of the modern "analogues" of Frege's symbols, this is to be taken with a pinch of salt. Roughly, we mean that the modern notions play similar roles in modern systems. In contrast, all functions, including the "truth-functions", for Frege have to be total, so, e.g., '$\top\ 0$' is not only well-formed, but in fact it is true in Frege's system (as an approximation, read "Zero is not the True"). This may give the reader a glimpse at the fact that Frege's system is quite different from modern predicate logic. Nonetheless, we append these modern analogues as "short hints" (to borrow Frege's phrase) that "are not exhaustive and make no claim to be of the strictest precision" (vol. I, p. 204, fn. 1). A somewhat more precise way of understanding what we intend by the modern "analogues" would go along the lines of: the restriction of Frege's functions (e.g., Frege's negation) to truth-values are co-extensive (under obvious translations of the rest of the language) with the cited modern truth-functions. (Thanks to an anonymous referee for pressing us on this point.)

[7]Compare fn 6 above.

Equality: $=$

The common equality relation; since for Frege sentences are names of truth-values, he also uses $=$ to express what in modern notation would be achieved by the biconditional: where p and q are propositions, "$p = q$" expresses that p and q are (or have) the same truth-value.

Quantifier: $-\!\overset{\mathfrak{a}}{\smile}\!-$

The universal quantifier with $\mathfrak{a}$ (set in *deutschen Buchstaben*, "German letters", actually a variant of Fraktur)[8] as the bound variable, analogous to $\forall\mathfrak{a}$.

Value-range operator: $\acute{\varepsilon}$

Variable-binding operator for the second-level function that maps first-level functions (including concepts) to their value-range; for instance, $\acute{\varepsilon}F(\varepsilon)$ is the extension that contains all and only those things that are F. More generally, for any function f, $\acute{\varepsilon}f(\varepsilon)$ is the *value-range* (essentially, the graph) of f. The bound variables use vowels from the Greek alphabet, and the accent on the variable is the Greek *spiritus lenis*, "smooth breathing", used to indicate the absence of aspiration in anterior vowels.

Analogue of the definite article: $\backslash\Delta$

Function that delivers the unique member of Δ if Δ is a singleton value-range (i.e., if Δ is $\acute{\varepsilon}(\varepsilon = t)$, for some term t), and Δ otherwise.

Roman letters: $p, q, \ldots$

Used for schematic generality, akin to free variables.

Frege describes the introduction of value-ranges as one of most consequential innovations in his system compared to the earlier version he presented in *Begriffsschrift* [19] (vol. 1, pp. IX–X) — although he did not fathom just how consequential it would prove to be.

It is also noteworthy that Frege's presentation of the conditional in two dimensional form:

$$\begin{array}{l} \ulcorner\, \Gamma \\ \llcorner\, \Delta \end{array}$$

led to a somewhat cumbersome presentation of his formulae (see Figure 1), for which he was criticised.[9] Frege noted as early as *Grundlagen* [20] that it is an important part of his logicism to state clearly everything on which a proven theorem rests. The two-dimensional framework is well-suited for that purpose: the often numerous subcomponents (antecedents) are stated below the supercomponent (consequent). Even the visual appearance suggests that the subcomponents support the supercomponent, or that the latter rests upon the former. Our figurative expressions in language appear to be represented graphically in the formalism. It has been suggested by Kreiser that Frege's notation, and the resulting two-dimensional appearance, may

[8]Rather similar (but not identical) to the *Luthersche Fraktur* (1708, J. E. Luther, Frankfurt), *Schul Fraktur* (1886, J. Schelter, C. F. Giesecke, Leipzig) and related faces.

[9]See [54, p. 415] for references and quotations from reviews of Frege's *Begriffsschrift*; Courbe writes of *notations bizarre* ("bizarre notations") and *symboles étranges* ("strange symbols") in his review of the first volume of *Grundgesetze* [11].

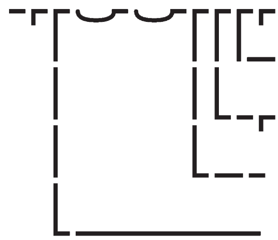

FIGURE 2. Pohle's construction of the parenthetical formula in Figure 1.

have been inspired by the work of the German philosopher Karl Christian Friedrich Krause [38, p. 164 ff].

The challenge of printing Frege's formulae was taken up by the typesetters of the Jena publisher Hermann Pohle (1845–1897). A close inspection of early printings usually reveals the method by which the concept-script formulae were constructed. As one would expect, only a small number of components are used, and these are standard wherever possible. In Figure 2 one can see the composition used by Pohle in vol. I, §158: upper and lower terminators of square parentheses and lines predominate.

§3. Function symbols. Throughout the text, Frege defines a series of functions with which to express the basic concepts of arithmetic. Frege emphasised the importance of these functions with an often startling typographic treatment: their symbols, imported from commerce, poetics, and phonetics, are rotated, shifted, and decorated with diacritics. Frege acknowledges in his foreword to *Grundgesetze*: "the first impression alone can only be off-putting: strange signs, pages of nothing but alien formulae" (vol. I, p. XI). However, Frege chose these mathematically unfamiliar symbols on principled grounds. As he explains in vol. II of *Grundgesetze*, new signs ought to be chosen for newly defined terms to ensure that the reader (and author!) does not rely on extraneous, previously associated content. For Frege, definitions have to be complete and must fully explain the newly introduced signs without recourse to informal notions, to intuition, or to any other source: "this is what logic requires; [...] reluctance to introduce new signs or words is the cause of many unclarities in mathematics" (vol. II, §58; see also [15, p. xxxi]). While the symbols Frege chose were unfamiliar in mathematics, he did pick them from the stock of Pohle's printshop, the Frommannsche Buchdruckerei (which specialised in printing scientific books and was located just down the street from Frege's house in Jena). Many of the symbols Frege decided to use appear to have been chosen for mnemonic reasons.

A confluence of some of the more cryptic of the functional symbolism can be found in the formula of vol. I, §122, shown in Figure 3, which defines the cardinal number *Endlos* ("Endless"), i.e., the first transfinite cardinal number, $\aleph_0$.[10]

[10] In his 1884 *Grundlagen* [20, §84], Frege still labelled this cardinal ∞_1.

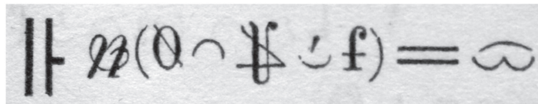

FIGURE 3. Concept-script of vol. I, §122 (enlarged).

The display starts with the double stroke of definition, ||⊢, centred on the baseline in contrast to the modern placement of the turnstile ($\vdash$) on the mathematics axis. The double stroke of definition is a combination of Frege's judgement stroke, ⊢, and a vertical line. In appearance, the latter is very similar to the German proofreading sign for "no indentation", while the double stroke of definition somewhat resembles that for "omitted line". The proofreading-signs have much longer horizontal strokes, however, and are in appearance closer to the way Frege's judgement-stroke appears in his earlier *Begriffsschrift* [19], ⊢—, published by L. Nebert (Halle). The German proofreading-signs were codified as norm DIN 16511 — but not until 1929, and thus long after the publication of *Grundgesetze*. In the late nineteenth century, a symbol resembling ⊐ was more commonly used to denote "no indentation" [3, vol. 10, p. 646].[11]

It seems most likely that ⊢, ||⊢, ┌ (from which the negation-stroke, ┬, is constructed, see Figure 2), —, etc., belong to a set of type that was used to construct frames for text-boxes of variable sizes and the above mentioned square parentheses (some word-processing software still contains symbols of this sort).

Next we have the cardinal number operator, ℔. The German word that Frege uses for "cardinal number" is *Anzahl*. The symbol resembles a cursive n, or perhaps a cursive A. Frege possibly chose it for this reason to stand for the Latin *numerus* or the German *Anzahl*. In fact it is an lb (pound) ligature, overturned. A variant on this form is included in Unicode (U+2114) and it can be seen in the 1933 specimen [36, p. 73], as well as in Dutch specimen books of the 18th century, and in French and Italian of the 19th (see [51]).

After an opening bracket and an occurrence of ϑ, which is to be discussed below, we have a small cap ⌒, sometimes called application for value-ranges: $a \frown \dot{\varepsilon} f(\varepsilon)$ delivers the same value as $f(a)$. In fact, Frege's theorem (1) (vol. I, p. 75) is $\vdash f(a) = a \frown \dot{\varepsilon} f(\varepsilon)$. For the special case of extensions, ⌒ essentially plays the role of membership. In this case theorem (1), tragically, becomes the exact analogue to naïve comprehension of extensions (and thus we are just one substitution step away from an explicit contradiction: take $\dot{\varepsilon}(\top\, \varepsilon \frown \varepsilon)$ for a, and let $f(\xi)$ be $\top\, (\xi \frown \xi)$). The symbol ⌒ seems to be a metrical breve (short syllable) overturned. Numerous metrical characters (i.e., those used to annotate metre in classical poetry, in particular the Greek) are present in the text: the cup-bar, ⏓, a metrical short over long (U+23D3), denoting "coupling" of two relations, i.e., an operation that collects pairs of objects from two relations to form a new relation; the cap-bar, ⌓, an overturned long

[11] Printshops usually had type for proofreading-signs in their stock. It is unknown what proofreading signs were available at Pohle's printshop.

over short (U+23D2), denotes a restriction of a relation; the bar-cap, ⌅, is the combination of relations, i.e., their union. (See Cook [9, pp. A-34–A-35] for more detailed explanations.)

The is used to represent the inverse of a relation (defined in vol. I, §39); the German term is *Umkehrung*. Again, it seems the symbol was chosen for its similarity with the letter U. Alternatively, a similarity to a cursive V could be observed, perhaps short for the Latin *vertatur* ("let it be inverted"), a well-known proofreading-sign that indicates the need to invert upside-down type. This , (U+F2F1),[12] is an old flourish currency sign for *Mark*, the German currency at that time. In variants of this character, the vertical line is joined to an "m" (see, for example, [51]).

Next, a small cup with an acute accent , employed to represent the weak ancestral of a relation. As previously, this is likely a metrical symbol (U+F706).[13] In addition, we have ⟂́ as a notation for the strong ancestral of a relation. Frege defines in terms of ⟂́ as follows: a stands in the weak ancestral of some relation to b if and only if a stands in the strong ancestral of that relation to b, or is identical to b. The signs were likely chosen for their obvious similarity.

The parenthetic expression in Figure 3 ends with the successor function ẝ: a medial (or long) s (U+017F) as seen in the left stroke of the Eszett, ß. Frege makes use of striking-out in distinguishing between the cardinal numbers 0̸, 1̸, . . . and the (real) numbers 0, 1, . . . (vol. I, §41). The successor acts on cardinal numbers and so we identify the cross-bar on the ẝ (which is distinctly on the mathematics-axis) as a striking-out, rather than the cross-bar of an f which would rest at x-height. Nonetheless, Frege possibly chose this symbol because of its resemblance with an f: the German word Frege uses to describe that a number *succeeds* another in the cardinal number series is *folgt* ("follows"). Alternatively, Frege might have chosen the medial s as short for the Latin *successor*.[14] The final symbol in Figure 3, ⏖, denotes the cardinal number *Endlos*, whose lower part has some resemblance to a lying cursive E. Distinct from the ∞ of John Wallis (which is also used, in vol. II, §§68, 81, 143, and 164, but only in examples and quotations from other mathematicians), ⏖ seems to be constructed from a lower part which is a joined pair of metrical shorts (U+23D6) (note the lack of ductus in the bowls) with a top-parenthesis placed above it. We note the resemblance to the closed omega ɷ used in the International Phonetic Alphabet (IPA). The ω was, of course, Cantor's sign for the first transfinite ordinal; Frege was

[12]This codepoint lies within the Private Use Area (PUA) of the Medieval Unicode Font Initiative (MUFI) Character Recommendation (version 3.0) and so is liable to change if adopted by the Unicode consortium.

[13]MUFI PUA, see fn 12.

[14]A more daring suggestion would be to regard Frege's choice as influenced by a well-known play on words in Goethe's *Faust II*. Goethe introduces a character called 𝔊𝔯𝔢𝔦𝔣 (*Greif*, Gryphon), whose dative plural, 𝔊𝔯𝔢𝔦𝔣𝔢𝔫, is easily misread as 𝔊𝔯𝔢𝔦ſ𝔢𝔫, (*Greisen*, dative plural of *Greis*, German for old man, with connotations of feebleness of mind and body; roughly: dotard). It is imaginable that Frege was inspired by Goethe's *calembour* and used ẝ to combine abbreviations of the Latin words *successor* and *functio*.

indeed familiar with Cantor's work. It is important to note, however, that in contrast to Wallis's ∞ and Cantor's ω, Frege's ∽ is a cardinal number, viz. $\aleph_0$.

Frege used numerous characters from the IPA. First published in 1888, it was the fruit of a century of development of phonetics. The Icelandic letter ð, adopted by the IPA for the voiced dental fricative (*th* in *than*), was used by Frege to define the domain of magnitudes (*Grössengebiet*). The IPA's overturned y, representing the palatal lateral approximant, is used underscored by Frege, ʎ̠, as seen in Figure 1. The ʎ̠-function cuts out a segment from a sequence, as it were. The symbol is introduced by Frege, as in "Γ⌢(Δ; Θ ʎ̠ Υ)", to state that

Γ belongs to the Υ-series running from Δ to Θ

(vol. I, §158). (Note that the semicolon denotes ordered pair.) The underscore diacritic indicates *retraction* (of the tongue) in phonetics.

Figure 1 displays another notable feature of Frege's symbolism: the unusual spacing. The small cap, ⌢, is set much looser outside the parenthesis than within. In most formulae with multiple subcomponents, subformulae containing ⌢, =, or other binary functions are spaced in such a way that the arguments align vertically across the subcomponents. A particularly striking example featuring inequalities is shown in Figure 4. Another oddity is that the pair function ";", although binary infix, binds much tighter to its left argument than its right.

In volume II of *Grundgesetze*, a number of function symbols are overturned roman lower-case letters with diacritics. In the first two frames of Figure 5 we see the overturned roman c and e with similar lachrymal hook diacritics. Their visual similarity is reflected in Frege's usage of them: the ə̉ denotes that an object, in a series starting with an object, is the first to belong to a given class; the ɔ̉ denotes that an object in a series is the first after an object to belong to a given class. The only difference between these functions is whether or not the first object is included. These characters are used in binary infix, presented at a smaller size (about a point smaller) than the body

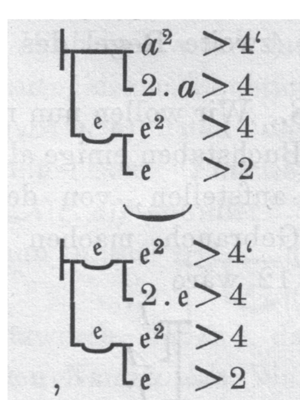

FIGURE 4. Alignment of inequalities in vol. I, §18.

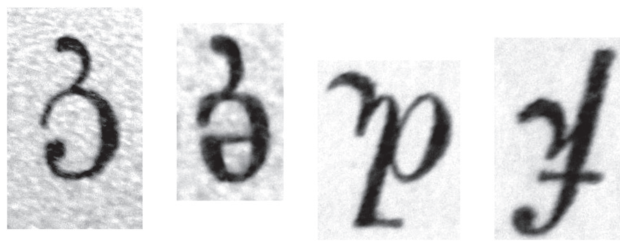

FIGURE 5. Overturned lower-case.

font and raised slightly above the baseline; possibly a hint to the reader as to their relational nature. As to the typographic origin of these characters, one can find an e with an *ogonek* as used to decorate vowels in Polish (indicating nasalisation) and Lithuanian (lengthening); one also has the *e caudata*, used in Latin writing as an alternate for the *ae* or *æ* vowel. A right-hooked c exists as the Cyrillic *small letter es with descender* (U+04AB) used in the Turkic Bashkir and Chuvash alphabets for the voiceless dental and alveolo-palatal fricatives, respectively. Yet in Frege's time the Bashkirians wrote in Arabic; and the Chuvashians adopted the Cyrillic alphabet (as modified by Russian missionaries) only in the 1870s. We remark also that the hooked e and c, as well as having no apparent common typographical lineage, also seem to be taken from different fonts — generally the lower part of the bowls of c and e are indistinguishable, but for Frege's characters, that of the (unrotated) e is squarer than that of the c, and its finial differs in thickness and stress. Yet these two have almost identical lachrymal diacritics (rather than the sharp hooks usually associated with the ogonek and the descenders of the decorated *es* and the *e caudata*).

The rightmost two frames of Figure 5 show the overturned italic d and f each with a hook attached top-left. (One can recognise that the overturned *d* is not a *p* from the one-sided serif at its foot — in fact this is the beak of the *d*.) Again, the common diacritic accompanies closely related denotation: The ɟ is a *Positivalklasse* (positival class) while ꝑ is a *Positivklasse* (positive class); every positive class is a positival class but not vice versa. Frege probably chose ꝑ for its resemblance with *p*, as short for *Positivklasse*. The diacritic (in the unrotated orientation) resembles that used in the IPA to denote *rhoticity* (a˞, ɛ˞, ə˞, . . .), whereby vowels are coloured by the r sound which follows, as in the North American pronunciation of *car* and *start*. There has been, as far as we are aware, no phonetic notation which has ever used f and d to denote vowels, so this observation leads nowhere. The origin of these characters remains an enigma.

Our (incomplete)[15] tour of the function symbols concludes with another mystery. Figure 6 shows what we have called the *left-angle*: $x; y \frown (A \nwarrow t)$

[15]Other function symbols include ⎵, a rotated square parenthesis, for composition of relations; ł, the *dark* (or *velarised*) l from the Polish alphabet, for limit; Ⅎ, an inverted italic F, for downward closure; and ꞵ, an inverted Eszett from the German alphabet, for the Archimedean property; see the complete list in [9].

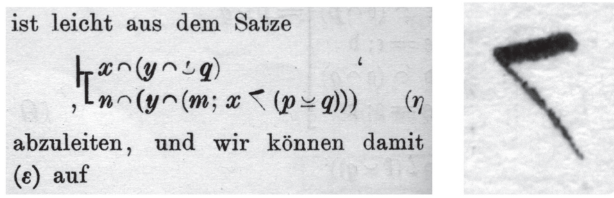

FIGURE 6. Concept-script of vol. I, §144, with detail of the left-angle $\nwarrow$.

holds of x and y when and only when the pair $x; y$ belongs to the series (of pairs) t beginning with (the pair) A. The glyph's stress suggests that it has been turned by an odd number of right angles, so one thinks of a modern surd-piece rotated clockwise (Peano used an overturned surd as shorthand for exponentiation [7, p. 301]), yet in Frege's time, surds were almost always presented with a beak at the end of the short stroke, and the slight curve of long stroke in the left-angle also argues against this interpretation. One finds a better candidate, it would seem, in the contemporary proofreading-sign *vertatur* (see, e.g., [55, figure on p. 169, and entry "Korrekturzeichen", p. 170]), again rotated by ninety degrees clockwise. It is not known however what correction-signs Pohle had available.[16]

§4. Other notation. While the function symbols of *Grundgesetze* provide a wealth of typographic variety, they by no means exhaust it; we mention a few of the other notational curiosities.

In Figure 7 we see what could easily be mistaken for a modern-day commutative diagram, if it were not for the delightfully representational fully-fletched arrows. In fact this diagram illustrates the construction of a mapping that Frege uses to prove that for any finite concept, the objects falling under it can be well-ordered. In the diagram, a, m, n, c, x, and y are objects, p and q relations; the lower up-arrow, $\uparrow$, labeled by $\smile q$, thus indicates that a stands in the weak ancestral of the q-relation to m (compare vol. I, §172). The delicate strokes of the fletching of these arrows are often incompletely inked or damaged, as can be seen in the detail of the figure.

The arrow is all the more curious given that the abstraction of arrows to their modern form, $\rightarrow$, had already taken place. While the most common uses of arrows in mathematical and logical notation all post-date Frege's work,[17] unfletched arrow signs had been common for quite some time

[16]See fn 11 above.

[17]For example: the arrow for limits, $\lim_{n\to\infty}$, apparently first used in 1905 by John Gaston Leathem [41], became widely accepted after 1908 with the publication of the books by Thomas John I'Anson Bromwich [4] and G. H. Hardy [31] (see [45]), and has been met "with enthusiastic adoption everywhere" [7, p. 339] (Weierstrass's notation was $\operatorname*{Lim}_{n=\infty}$, which Frege adopts in *Grundgesetze*); the vector notation, $\vec{a}$, appears to be the invention of Paul Langevin in 1912 [1]; the use of the arrow in functional representation first appears, according to Mac Lane [42, p. 29], in the 1940s with Hurewicz [37]; the use of $\rightarrow$ as an alternative to Peano's $\supset$ for the material conditional was first used in 1922 by Hilbert [35].

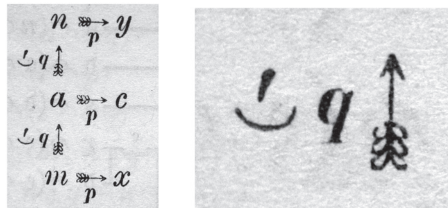

FIGURE 7. Diagram with fletched arrows vol. I, §172 and detail.

(see [6, 48], for example) and they were certainly widely used in diagrams. Frege, in *Grundgesetze*, was not the last to cling to this anachronism. A very similar arrow (but with a more robust, solid fletching) was used in the influential first series of the journal *Rendiconti del Circolo Matematico di Palermo*, for example in Hardy and Littlewood's only paper for the journal, [30] in 1916, and as late as 1933 in a paper by Romanovsky [46] (the former being more accessible since it can be found in Hardy's collected papers [29, p. 609]). A similar fletched arrow also occurs in a work by Johannes Thomae [52, p. 109], who was Frege's colleague in Jena and also one of Frege's favorite targets for his criticism and biting polemics (e.g., [23, §§86–137], [24]).

In vol. II, §83, Frege uses a variety of lazy capitals and numerals in his criticism of Georg Cantor's definition of the arithmetical operations (Figure 8). Frege replaces $<$, $>$, and $=$ by ∾, ∿, and ∽, respectively, and $+$, $-$, and . by ⩋, ⩊, and ∪, respectively, in Cantor's definition of these items. Similarly, he replaces familiar words for mathematical operations and concepts (equal, greater than, less than, sum, difference, zero, etc.) by nonsense words (azig, bezig, zezig, arung, berung, poll, etc.).[18] Any influence of prior knowledge of the subject matter is thereby ruled out, and Frege argues that the sense conveyed upon these unknown words and symbols by Cantor's explanation is not sufficiently precise to serve as a proper definition of these notions. The text in Figure 8 translates as:

> Now finally, we come to the definitions of being azig, bezig and zezig, between two numbers, b and b'; namely, we say that b ∽ b' or b ∾ b' or b ∿ b' depending on whether b ⩊ b' is azig poll or bezig poll or zezig poll.

As Frege observes, "the sense thus conferred on these new words and signs does not suffice for the purpose at hand" [28, vol. II, §83].

We are thus here taken back to Frege's methodological reason for the adoption of unfamiliar symbols in his own development of arithmetic that we mentioned above (§3). In Frege's view, the advantage of using unfamiliar signs lies in ensuring that no prior understanding of the notions that are to be defined creates the illusion that a given — actually insufficient — definition

[18] *-ig* is a German word-ending for adjectives, *-ung* for nouns; *a*, *be*, *ze* are chosen for the German pronunciation of the letters at the beginning of the alphabet: A, B, C. *Poll* appears to be a flight of fancy.

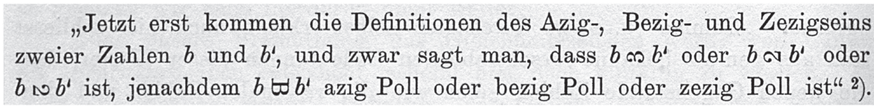
„Jetzt erst kommen die Definitionen des Azig-, Bezig- und Zezigseins zweier Zahlen b und b', und zwar sagt man, dass $b \backsim b'$ oder $b \backsim b'$ oder $b \backsim b'$ ist, jenachdem $b \uplus b'$ azig Poll oder bezig Poll oder zezig Poll ist“ ²).

FIGURE 8. Lazy capitals and numerals from vol. II, §83.

exhaustively characterises its definiendum; that is, if we use "newly created words and signs with which neither a sense nor the appearance of a sense is already associated" [28, vol. II, §83] in place of the familiar signs and common words that Cantor uses, we see that no proper meaning has been established for these by means of Cantor's explanations.[19]

§5. Reception and legacy. According to Cajori, Frege's early neglect "has been attributed to Frege's repulsive symbolism" [7, p. 295]. Cajori's obvious distaste coincides with an uncharacteristic sloppiness in the reproduction of the notation in his text. The *Begriffsschrift* is set on the mathematics axis rather than the baseline; the universally quantified variable is set in roman rather than German face, and in the example of p. 297, the first $\mathfrak{a}$ in $\top\!\overset{\mathfrak{a}}{\smile}\, 2 + 3.\mathfrak{a} = 5\mathfrak{a}$, is set so small as to be a dot.

But perhaps we can accept the historical judgement without the aesthetic. Frege's notational legacy consists of the *Begriffsschrift* strokes of judgement (assertion) and negation; both lifted from the baseline and lightened to the colour of the arithmetic operators, the negation also losing its snout. The defined function symbols of *Grundgesetze*, and of *Begriffsschrift* have not been adopted.

In [26, §13], Bynum relates the difficulty that Frege had in finding a publisher for the *Grundgesetze*, eventually persuading Hermann Pohle to take up the project in two volumes, the publication of the second contingent on the reception of the first. However, given the poor sales, Frege paid for the publication of the second volume out of his own pocket.

The title of this paper refers to a phrase used by Frege in his article [22], comparing his own notation with that of Peano. In a spirited defence of the two-dimensionality of the *Begriffsschrift* he asserts

[19]It is one of the great tragedies of the history and philosophy of mathematics that two of the giants of foundational studies, Frege and Cantor, did not engage seriously with each other's accounts. Frege does not discuss Cantor's great achievements appropriately, but merely quarrels over the point described above in *Grundgesetze*; Cantor, in turn, gave Frege only the most cursory of readings, which lead to two grave misunderstandings in Cantor's review of Frege's *Grundlagen* (see [14]): Cantor misunderstands Frege's definition of cardinal number (and bases part of his criticism on this erroneous understanding); moreover, he fails to notice that when Frege uses the word *Anzahlen* he means (and defines them as) *cardinal* numbers. Cantor assumes that Frege, like Cantor himself, means *ordinal* number by *Anzahl*, despite the fact that Frege explicitly points out this difference in terminology in *Grundlagen* [20, §85]. Frege does however have some admiration for Cantor's 1883 "Grundlagen einer allgemeinen Mannigfaltigkeitslehre" [8] calling it "a remarkable work" in this earlier book [20, ibid.].

> In the Peano concept-script the presentation of formulas upon a single line has apparently been accomplished in principle. To me this seems a gratuitous renunciation of one of the main advantages of the written over the spoken. After all, the convenience of the typesetter is not the *summum bonum*.

In a sense, the diagrams of Young and Feynman and the commutative diagrams of homological algebra make Frege's point, but surely these do not demand the kind of typographic efforts that Frege sought from Pohle.

In recent years there have been major developments in the study of Frege's logicism. Since the discovery of what is now called *Frege's Theorem* [33, 58] — the proof that the axioms of arithmetic can be derived in second-order logic using Hume's Principle (a principle governing the identity of cardinal numbers and derived by Frege from Basic Law V) and Frege's definition of zero, predecession and natural number — there has been a revival of the logicist idea often under the heading *neo-Fregeanism* or *neo-logicism*, in both philosophy and mathematics. Removing value-ranges (and Basic Laws V and VI, which feature them), Frege's system is essentially standard second-order logic, and it has also been shown that fragments of the *Grundgesetze* system using predicative versions of Basic Law V are consistent [2, 5, 18, 32, 56]. This recent revival has also lead to a closer study of Frege's original writings (e.g., [34]) and lead to the first full translation of Frege's magnum opus: *Grundgesetze der Arithmetik* into English [28].

§**6. Acknowledgments.** The authors wish to thank L. A. Seagrave, for preparing the photographs; Andreas Stötzner, Leipzig (Deputy Chair, MUFI), and Christophor Kozhuharov for numerous helpful comments and corrections; Udo Wermuth, for the identification of the Mark symbol, ℳ; W. Zainer, for the reference [36]; and an anonymous referee for helpful comments. Philip Ebert would like to acknowledge support from the Arts and Humanities Research Council (AH/J00233X/1) while working on this manuscript.

REFERENCES

[1] M. Abraham and Paul Langevin, *Notions géométriques fondamentales*, ***Encyclopédie des sciences mathématiques pures et appliquées,*** tome IV, vol. 5, Gauthier-Villars, Paris, 1912, pp. 1–60.

[2] Francesca Boccuni, *Plural* Grundgesetze. ***Studia Logica***, vol. 96 (2010), pp. 315–330.

[3] ***Brockhaus' Konversations-Lexikon***, 14^{th} ed., vol. 10, Brockhaus, Leipzig, 1894.

[4] Thomas John I'Anson Bromwich, ***An introduction to the theory of infinite series***, 1 ed., Macmillan, London, 1908.

[5] John P. Burgess, ***Fixing Frege***, Princeton Monographs in Philosophy, Princeton University Press, Princeton, NJ, 2005.

[6] Oliver Byrne, ***Dual arithmetic: A new art***, Bell and Daldy, London, 1863.

[7] Florian Cajori, ***A history of mathematical notations***, vol. II, Open Court, Chicago, 1952.

[8] Georg Cantor, *Ueber unendliche, lineare Punktmannichfaltigkeiten.* 5. Fortsetzung: *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, ***Mathematische Annalen***, vol. XXI (1883), pp. 545–590. English translation by William Ewald in [17], vol. II, pp. 881–920.

[9] Roy T. Cook, *How to read* Grundgesetze, [28], pp. A1–A42 (appendix).

[10] ———, *Frege's little theorem and Frege's way out*, ***Essays on Frege's* Basic Laws of Arithmetic** (Philip A. Ebert and Marcus Rossberg, editors), Oxford University Press, Oxford, 2014.

[11] H. Courbe, *Review of* [21], ***Polybiblion: Revue bibliographique universelle. Partie Littéraire.*** Deuxième Série. Tome Quarantième, vol. 71 (1894), pp. 428–429. English translation by Charlotte A. Geniez in Ebert and Rossberg [16].

[12] Michael Dummett, *Frege's way out: A footnote to a footnote*. ***Analysis***, vol. 33 (1971), pp. 139–140.

[13] ———, ***Frege: Philosophy of language***, 2nd ed., Harvard University Press, Cambridge, MA, 1981.

[14] Philip A. Ebert and Marcus Rossberg, *Cantor on Frege's* Foundations of Arithmetic*: Cantor's 1885 review of Frege's* Die Grundlagen der Arithmetik. ***History and Philosophy of Logic***, vol. 30 (2009), pp. 341–348.

[15] ———, *Translators' introduction*, [28], pp. xiii–xxxix.

[16] Philip A. Ebert and Marcus Rossberg (editors), ***Essays on Frege's* Basic Laws of Arithmetic**, Oxford University Press, Oxford, forthcoming.

[17] William Bragg Ewald (editor), ***From Kant to Hilbert: A source book in the foundations of mathematics.*** 2 vols., Oxford University Press, Oxford, 1996.

[18] Fernando Ferreira and Kai F. Wehmeier, *On the consistency of the Δ^1_1-CA fragment of Frege's* Grundgesetze. ***Journal of Philosophical Logic***, vol. 31 (2002), no. 4, pp. 301–311.

[19] Gottlob Frege, ***Begriffsschrift: Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens***, Verlag L. Nebert, Halle a. d. Saale, 1879, English translation by S. Bauer-Mengelberg in [53], pp. 1–82; and by T. W. Bynum in [26].

[20] ———, ***Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl***, Wilhelm Koebner, Breslau, 1884, English translation: [25].

[21] ———, ***Grundgesetze der Arithmetik***, vol. 1, Hermann Pohl, Jena, 1893, English translation in [28].

[22] ———, *Ueber die Begriffsschrift des Herrn Peano und meine eigene*, ***Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physikalische Klasse***, vol. XLVIII (1897), pp. 361–378, English translation in [27].

[23] ———, ***Grundgesetze der Arithmetik***, vol. 2, Hermann Pohl, Jena, 1903, English translation in [28].

[24] ———, *Antwort auf die Ferienplauderei des Herrn Thomae*. ***Jahresbericht der Deutschen Mathematiker-Vereinigung***, vol. 15 (1906), pp. 586–590. English translation in [27], pp. 341–345.

[25] ———, ***The foundations of arithmetic***. Translated by J. L. Austin, Blackwell, Oxford, 1950.

[26] ———, ***Conceptual notation and related articles.*** Ed. and trans. by T. W. Bynum, Oxford University Press, Oxford, 1972.

[27] ———, ***Collected papers on mathematics, logic, and philosophy.*** Ed. by B. McGuinness, Blackwell, Oxford, 1984.

[28] ———, ***Basic laws of arithmetic.*** Ed. and trans. by P. A. Ebert and M. Rossberg, Oxford University Press, Oxford, 2013.

[29] G. H. Hardy, ***Collected papers of G. H. Hardy: Including joint papers with J. E. Littlewood***, vol. VI, Clarendon Press, Oxford, 1974.

[30] G. H. Hardy and J. E. Littlewood, *Theorems concerning the summability of series by Borel's exponential method*, ***Rendiconti del Circolo Matematico di Palermo,*** First Series, vol. 41 (1916), pp. 36–53, Reprinted in [29, p. 609].

[31] Godfrey Harold Hardy, ***A course of pure mathematics***, University of Michigan Library, Ann Arbor, 1908.

[32] Richard G. Heck, Jr, *The consistency of predicative fragments of Frege's* Grundgesetze der Arithmetic. ***History and Philosophy of Logic***, vol. 17 (1996), pp. 209–220.

[33] ———, ***Frege's theorem***, Oxford University Press, Oxford, 2011.

[34] ———, ***Reading Frege's* Grundgesetze**, Oxford University Press, Oxford, 2012.

[35] David Hilbert, *Neubegründung der Mathematik*. ***Abhandlungen aus dem Mathematischen Seminar der Hamburger Universität***, vol. 1 (1922), pp. 157–177.

[36] J. P. Himmer, ***Unsere Schriften/Buchdruckerei***, Himmer, Augsburg, 1933. Printer's specimen, copies at the Bayerische Staatsbibliothek, Munich and at the Staats- und Stadtbibliothek, Augsburg.

[37] W. Hurewicz, *On duality theorems*. ***American Mathematical Society, Bulletin***, vol. 47 (1941), pp. 562–563.

[38] Lothar Kreiser, ***Gottlob Frege: Leben, Werk, Zeit***, Meiner Felix Verlag, Hamburg, 2001.

[39] Gregory Landini, *The ins and outs of Frege's way out*. ***Philosophia Mathematica***, vol. 14 (2006), pp. 1–25.

[40] ———, ***Frege's notations: What they are and how they mean***, Palgrave Macmillan, New York, 2012.

[41] John Gaston Leathem, *Volume and surface integrals used in physics*, ***Cambridge Tracts in Mathematics and Mathematical Physics***, vol. 1, Cambridge University Press, Cambridge, 1905.

[42] Saunders Mac Lane, ***Categories for the working mathematician***, Springer, New York, 1971.

[43] W. V. Quine, ***Methods of logic***, Henry Holt, New York, 1950.

[44] ———, *On Frege's way out*. ***Mind***, vol. 64 (1955), pp. 145–159.

[45] Dave L. Renfro, *The arrow notation for limits*, `http://jeff560.tripod.com/calculus.html`, 2010.

[46] V. Romanovsky, *Sur une généralisation de la loi sinusoïdale limite*. ***Rendiconti del Circolo Matematico di Palermo,*** First Series, vol. 57 (1933), pp. 130–136.

[47] Bertrand Russell, ***Letter to Frege***, In van Heijenoort [53], pp. 124–125.

[48] Schellbach, *Über die Zeichen der Mathematik*. ***Journal für die reine und angewandte Mathematik (Crelle's Journal)***, vol. XII (1834), no. 1–2, pp. 70–81, 148–166.

[49] Boleław Sobociński, *L'analyse de l'antinomie russellienne par Leśniewski*, ***Methodos***, vol. 1–2 (1949-1950), pp. 94–107, 220–228, 308–316; 6–7, 237–257. English translation by Robert E. Clay: [50].

[50] ———, *Leśniewski's Analysis of Russell's Paradox*, ***Leśniewski's systems: Ontology and mereology*** (J. T. J. Srzednicki, V. F. Rickey, and J. Czelakowski, editors), Martinus Nijhoff, Boston, 1984, pp. 11–44.

[51] Andreas Stötzner, *Gewichts- und Währungszeichen: Übersicht,* Auszug aus ***Signa: Beiträge zur Signographie*** 3 (2002): Zweite ergänzte Fassung 1.2, 2005.

[52] J. Thomae, ***Elementare Theorie der analytischen Functionen einer complexen Veränderlichen***, 2nd ed., Nebert, Halle a. d. Saale, 1898.

[53] Jean van Heijenoort (editor), ***From Frege to Gödel: A source book in mathematical logic 1879–1931***, Harvard University Press, Cambridge, MA, 1967.

[54] Risto Vilkko, *The Reception of Frege's* Begriffsschrift. ***Historia Mathematica***, vol. 25 (1998), pp. 412–422, article no. HM982213.

[55] Alexander Waldow (editor), ***Illustrierte Encyklopädie der graphischen Künste und der verwandten Zweige***, Druck und Verlag von Alexander Waldow, Leipzig, 1884.

[56] Kai F. Wehmeier, *Consistent fragments of* Grundgesetze *and the existence of non-logical objects*. ***Synthese***, vol. 121 (1999), pp. 309–328.

[57] Kai F. Wehmeier and H.-C. Schmidt am Busch, *The quest for Frege's Nachlass*, ***Critical assessments of leading philosophers: Gottlob Frege*** (Michael Beaney and Erich Reck, editors), vol. 1, Routledge, London, 2005, pp. 54–67.

[58] Crispin Wright, ***Frege's conception of numbers as objects***, Scots Philosophical Monograph Series, no. 2, Aberdeen University Press, Aberdeen, 1983.

WANSTEAD
LONDON, UK
E-mail: j.j.green@gmx.co.uk

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF CONNECTICUT
USA
E-mail: marcus.rossberg@uconn.edu

DIVISION OF LAW AND PHILOSOPHY
UNIVERSITY OF STIRLING
SCOTLAND, UK
E-mail: p.a.ebert@stir.ac.uk